Dynamical Systems

Introduction

In general the term dynamical systems refer to the equations (difference and /or differential) governing the time varying behavior of some physical systems. In fact, the theory of dynamical systems is concerned with the evolution of the systems (i.e change of the system state in time) as parameters which control the system, and initial state of the system are varied.
A natural way to describe evolution of the system is to consider the map

$$x : T \to X \Rightarrow t \to x(t) \qquad (1)$$

Here $x(t)$ can be described as the state of the system at time t.

Remarks:

(1) Since the system can only be in one state at any moment of time, the map x is a single valued function such a function determines the present state in terms of past state.

(2) It is required that, the function (or map) given by Eq. (1) be deterministic no randomness is allowed, which means that we can determine the present state uniquely from the past state.

Now,

If $T = R_+$ or $T = R$, then T is known as a continuous time.

If $T = Z_+$ or $T = Z$, then T is known as a discrete time.

Furthermore, X is said to be the state space of the system

Definition:
The state space of the system X is defined to be the set of all conceivable states of a system.
Accordingly, if $X \subseteq R^n$ $(n \in N)$ and $x \in X$ then $x = (x_1, x_2, \dots x_n)^T$ with $x_i \in R$ and $(i = 1, 2, \dots, n)$.

Types of Dynamical Systems:
There are two types of dynamical systems

1- Continuous time dynamical system
in which the map is applied at a continuous time, further the structure of the system described by a differential equation or Vector Veiled.

2- Discrete time dynamical system
in which the map is applied iteratively at discrete times. Further the structure of the system described by difference equations.

Definition
Consider the following Vector field or differential equation

$$\frac{dx}{dt} = f(x), \quad x(0) = x_0 \in R^p \qquad (2)$$

Where $f \in C(R^p, R^p)$. Then the triple $\{T, X, f\}$ is said to define a continuous time dynamical system on a subset $E \subseteq R^p$ if for every $x \in E$ there exists

a unique solution $x(t, x_0)$ of Eq. (2) which is defined for all $t \in [0, \infty)$ and remaining in E for all $t \in [0, \infty)$.

Definition:
The continuous differentiable function $x: I \to R^p$, which is satisfying eq. (2) where I is an interval in R containing the origin, is called a solution of equation (2).

Classification of differential equation with respect to time

1. If the function f in equation (2) does not depends on time explicitly, then the differential equation given by (2) is called an autonomous equation.

2. If $f = f(x, t)$ in equation (2), that is mean the function f depends explicity on time then equation (2) can berewritten as

$$\frac{dx}{dt} = f(x, t), \quad x(t_0) = x_0 \in R^p \qquad (3)$$

Then equation (3) is called non-autonomous equation

Notation: $\dot{x} = \frac{dx}{dt} = x'(t)$

It is well known that, n^{th} order autonomous differential equation may be written in the form of system consisting of nth first order differential equations. However, the nth order non-autonomous differential equation may be written in the form of system consisting of $(n+1)^{th}$ first order differential equations.

Maps Vs. Difference Equations

Definition:

Consider a map $f: R \to R$ where R is the set of real numbers. Then the orbit $\mathcal{O}(x_0)$ of a point $x_0 \in R$ is defined to be the set of points

$$\mathcal{O}(x_0) = \{x_0, f(x_0), f^2(x_0), f^3(x_0), \ldots\}$$

where $f^2 = f \circ f$, $f^3 = f \circ f \circ f$, etc.

If we let $x(n) = f^n(x_0)$, then we obtain the first-order difference equation

$$x(n+1) = f(x(n)) \qquad (4)$$

with $x(0) = x_0$

In population biology, $x(n)$ may represent a population size in generation n. Eq. (4) simply states that the size $x(n+1)$ of a population in generation $n+1$ is related to the size $x(n)$ of the population in the preceding generation n by the function f.

In epidemiology, $x(n)$ represents the fraction of the population infected at time n.

In economics, $x(n)$ may be the price per unit in time n for a certain commodity.

In the social sciences, $x(n)$ may be the number of bits of information that can be remembered after a period n.

Ex Let $f(x) = x^2$, then

$O(1) = \{1\} \longrightarrow 1$

$O(2) = \{2, 4, 16, \dots\} \longrightarrow \infty$

$O(\frac{1}{2}) = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \dots\} \longrightarrow 0$

$O(\frac{-3}{4}) = \{\frac{-3}{4}, \frac{9}{16}, \frac{81}{256}, \dots\} \longrightarrow 0$

We can note that , $O(\frac{-3}{4})$, $O(0)$, $O(\frac{1}{2})$ attracting to 0

Ex $f(x) = 3x(1-x)$, So

$O(\frac{1}{2}) = \{\underset{x_0}{\frac{1}{2}}, \underset{f}{\frac{3}{4}}, \underset{f^2}{0.5625}, \underset{f^3}{0.7383}, \dots, \underset{f^{40}}{0.63}, \dots\}$

$O(1) = \{1, 0\} \longrightarrow 0$

$O(\frac{1}{4}) = \{\frac{1}{4}, 0.5625, 0.7383, \dots, 0.6781, \dots\}$

$O(0) = \{0\}$

Note that $O(\frac{1}{2})$, $O(\frac{1}{4})$ attracting to $\frac{2}{3}$ and repelling from 0

Remark

In $O(x_0)$ may be use $x_1, x_2, \dots$ we mean that $f(x_0) = x_1$, $f^2(x_0) = x_2$, ---

idea Plus

Definition

A point $p \in Dom_f$ is said to be a fixed point (equilibrium point) (steady state) of f if $f(p) = p$

Ex for $f(x) = x^2$, we have two fixed points

$$p_0 = 0 \quad , \quad p_1 = 1$$

for $f(x) = 3x(1-x)$ we have

$$p_0 = 0 \quad , \quad p_1 = \frac{2}{3}$$

To find all equilibrium points, we must solve the equation $f(x) = x$.

Graphically speaking, a fixed point of a map f is a point where the curve $y = f(x)$ intersects the diagonal line $y = x$.

Note that the above definition of fixed point for discrete dynamical systems (i.e we have difference equation ΔE)

$$X(n+1) = f(X(n))$$

or for simplicity

$$X_{n+1} = f(X_n) \quad , n \text{ represent the time}$$

While the definition of fixed point for continuous dynamical systems (i.e we have differential equation DE)

$$x'(t) = \frac{dx}{dt} = f(X(t)) \quad , X(0) = X_0$$

is a point which satisfy $f(p) = 0$

idea That is mean $X'(t) = 0 \rightarrow X(t) = C$

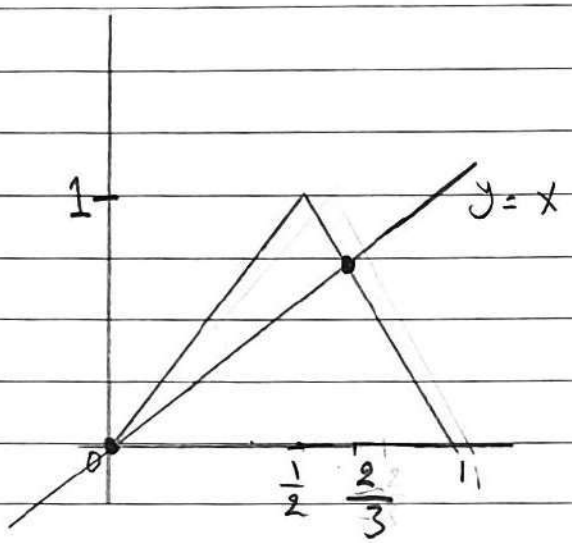
Eample. The Tent map

$$T(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2} \\ 2(1-x) & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

Find all fixed points and draw the graph of T.

Sol. $2x = x \implies x = 0 = p_0$

$2(1-x) = x \implies x = \frac{2}{3} = p_1$



Clear that $T : [0,1] \longrightarrow [0,1]$
and T has max. at $x = \frac{1}{2}$, $T(\frac{1}{2}) = 1$

Definition: one dimensional dynamical System is a function (map) $f : I \longrightarrow I$, where I is some interval in R.

Examples.

① $f(x) = x^2$ on $\mathbb{R}$ is a dynamical system ($x_{n+1} = x_n^2$ DE)

② $f(x) = x^2$ on $[-2, 2]$ is not dynamical system
since $f(2) = 4 \notin [-2, 2]$

③ T (the tent map) is a dynamical system on $[0, 1]$.

④ $L_2(x) = 2x(1-x)$ on $[0, 1]$ is a dynamical system.

Maps Vs. Differential Equations
Euler's Method

Consider the differential equation

$$x'(t) = g(x(t)) \quad , x(0) = x_0 \tag{5}$$

where $x'(t) = \frac{dx}{dt}$

For many differential equations such as Equation (5), it may not be possible to find a "closed form" solution. In this case, we resort to numerical methods to approximate the solution of Eq. (5). In the Euler algorithm, for example, we start with a discrete set of points $t_0, t_1, \ldots, t_n, \ldots$, with $h = t_{n+1} - t_n$ as the step size. Then for $t_n \leq t < t_{n+1}$ we approximate $x(t)$ by $x(t_n)$ and $x'(t)$ by $\frac{x(t_{n+1}) - x(t_n)}{h}$. Equation (5) now yields the difference equation

$$x(t_{n+1}) = x(t_n) + h\, g(x(t_n)) \tag{6}$$

which may be written in the simpler form

$$x(n+1) = x(n) + h\, g(x(n)) \tag{6}$$

where $x(n) = x(t_n)$

Note that Equation (6) is of the form of Equation (4) with

$$f(x) = f(x,h) = x + h\, g(x)$$

Now given the initial data $x(0) = x_0$, we may use Equation (6) to generate the values $x(1), x(2), x(3), \ldots$ These values approximate the solution of the differential Equation (5) at the "grid" points $t_1, t_2, t_3, \ldots$ provided that h is sufficiently small.

Example: Let $x'(t) = 0.7\,x^2(t) + 0.7$ D.E

$x(0) = 1$, $t \in [0,1]$

Apply Euler's method

$$x_{n+1} = x_n + h[0.7\,x_n^2 + 0.7]$$ D E

where $h = t_{i+1} - t_i$

Example. (The Logistic map)

$L_M(x) = Mx(1-x)$ Logistic map

$x_{n+1} = M x_n (1 - x_n)$ Logistic equation

$L_M(x) = Mx - Mx^2$ $M > 0$ $L_M : I = [0,1] \to I$

$L'_M(x) = M - 2Mx$

$M - 2Mx = 0$

$\to x = \frac{1}{2}$ is the only critical point

$(\frac{1}{2}, \frac{M}{4})$ max point

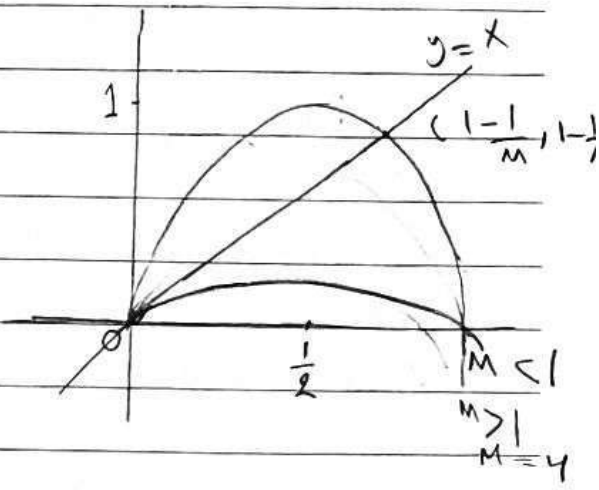
$M \in (0,4]$, $L_M : [0,1] \to [0,1]$

The Fixed points of L_M

$$Mx(1-x) = x$$
$$Mx - Mx^2 = x$$
$$Mx^2 + x - Mx = 0$$
$$x[Mx + (1-M)] = 0$$

$$x = 0 = p_0$$

$$x = \frac{M-1}{M} = 1 - \frac{1}{M} = p_1$$



Clear that

If $M = 1$ then $p_1 = p_0 = 0$

If $M < 1$ then only one fixed point $p_0 = 0$

If $M > 1$ then

L_M have two fixed points p_0, p_1

For example if $M = 2$, we have

$L_2(x) = 2x(1-x)$, $p_0 = 0$, $p_1 = \frac{1}{2}$ (Fixed points)

Remark: If we have D.E

$$x'(t) = f(x(t)), \quad x(0) = x_0 \qquad \text{DE}$$

So by Euler's method

$$x_{n+1} = x_n + h f(x_n) \qquad \text{DE}$$

to find the fixed points of DE

$x + h f(x) = x$

$\therefore h f(x) = 0$, but $h \neq 0$

$\therefore f(x) = 0$ is the fixed point of D.E

Example of D E

$x_{n+1} = a x_n$ is a linear first order difference equation (D E) , i.e the map is $f(x) = ax$, a is constant , the solution of this D E is

$x_n = a^n x_0$, $n \in Z^+$

if $|a| > 1$ $|x_n| \longrightarrow \infty$

if $|a| < 1$ $|x_n| \longrightarrow 0$

$a = 1$

$a = -1$

Definition: (Eventually fixed point)

A point x is said to be an eventually fixed point of a map f if there exists a positive integer r and a fixed point p of f such that

$$f^r(x) = p \quad \text{but} \quad f^{r-1}(x) \neq p$$

We denote the set of all fixed points by $Fix(f)$, the set of all eventually fixed points by $EFix(f)$ and the set of all eventually fixed points of the fixed points p by $EFix_p(f)$.

$$Fix(f) = \{ x : f(x) = x \}$$
$$EFix(f) = \{ x : f^n(x) = p, n \in Z^+, f(p) = p \}$$
$$EFix_p(f) = \{ x : f^n(x) = p, \text{for given fixed point } p, n \in Z^+ \}$$

Example

Consider the logistic map $f(x) = 2x(1-x)$

Then there are two fixed points

$p_0 = 0$, $p_1 = \frac{1}{2}$

To find ① $EFix_{p_0}(f)$, ② $EFix_{p_1}(f)$

First we need to find $f^{-1}(x)$

$$x = 2y(1-y)$$

So $x = 2y - 2y^2 \rightarrow 2y^2 - 2y + x = 0$

$$y = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{2 \pm \sqrt{4-4(2)x}}{2(2)}$$

$$= \frac{2 \pm \sqrt{4-8x}}{4} = \frac{1}{2} \pm \frac{\sqrt{1-2x}}{2}$$

$$\therefore y = \frac{1}{2}\left[1 \pm \sqrt{1-2x}\right] = f^{-1}(x)$$

Plus idea

① Thus $f^{-1}(0) = \{0, 1\}$

$f^{-1}(1)$ not exist

So $EFix_{p_0}(f) = \{0, 1\}$

$f^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}$

② $f^{-1}\left(\frac{1}{2}\right) = \left\{\frac{1}{2}\right\}$, So $EFix_{p_1}(f) = \left\{\frac{1}{2}\right\}$

Let us now contemplate a more interesting example

$L_M(x) = Mx(1-x)$ the Logistic map for $M = 4$
$L_4(x) = 4x(1-x)$

Find $EFix_p$ for each p fixed point of L_4

Sol First of all we have
$p_0 = 0$ and $p_1 = 1 - \frac{1}{M} = 1 - \frac{1}{4} = \frac{3}{4}$

Now want to find $L_4^{-1}(x)$, so

$y = 4x(1-x) = 4x - 4x^2$

$x = 4y - 4y^2 \Rightarrow 4y^2 - 4y + x = 0$

Solve for y

$$y = \frac{4 \pm \sqrt{16 - 16x}}{8} = \frac{1}{2}\left[1 \pm \sqrt{1-x}\right]$$

For $p_0 = 0$,

$L_4^{-1}(0) = \{0, 1\}$, $L_4^{-1}(1) = \frac{1}{2}$

$L_4^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}\left[1 \pm \sqrt{\frac{1}{2}}\right]$ ...., So there are ainfinitly many eventually fixed points.

For $p_1 = \frac{3}{4}$

$L_4^{-1}\left(\frac{3}{4}\right) = \frac{1}{2}\left[1 \pm \sqrt{1 - \frac{3}{4}}\right] = \frac{1}{2}\left[1 \pm \frac{1}{2}\right]$

$= \left\{\frac{3}{4}, \frac{1}{4}\right\}$

$L_4^{-1}\left(\frac{1}{4}\right) = \frac{1}{2}\left[1 \pm \sqrt{1 - \frac{1}{4}}\right] = \frac{1}{2}\left[1 \pm \frac{\sqrt{3}}{2}\right]$

$= \{0.067, 0.933\}$ ...

So the set $EFix_{p_1}(L_4)$ is infinite.

Example: Find the difference equation from the Logistic equation

$$x'(t) = \lambda x(t)(1 - x(t)) \qquad \text{DE}$$

Sol. By Euler's method

$$x_{n+1} = x_n + h f(x_n)$$

$$y_{n+1} = y_n + h f(y_n)$$
$$= y_n + h\lambda y_n (1 - y_n)$$

$$= y_n [1 + h\lambda (1 - y_n)]$$

$$= y_n [1 + h\lambda - h\lambda y_n]$$

$$\therefore y_{n+1} = (1 + h\lambda) y_n \left[1 - \frac{h\lambda}{1 + h\lambda} y_n\right]$$

Set $x_n = \frac{h\lambda}{1 + h\lambda} y_n$

So $y_n = \frac{1 + h\lambda}{h\lambda} x_n$

$$\frac{1 + h\lambda}{h\lambda} x_{n+1} = (1 + h\lambda) \frac{1 + h\lambda}{h\lambda} x_n (1 - x_n),$$

$$\therefore x_{n+1} = (1 + h\lambda) x_n (1 - x_n)$$

Set $M = 1 + h\lambda \Rightarrow x_{n+1} = M x_n (1 - x_n)$ DE

So the logistic map is

$$f(x) = Mx(1 - x)$$

idea Plus

Theorem (Brouwer fixed point theorem) (1910)
(1881–1966)

Let $f: I \to I$ be a continuous map, where $I = [a,b]$ is a closed interval in $\mathbb{R}$. Then f has a fixed point.

proof

Define $g(x) = f(x) - x$

Then $g(x)$ is also continuous map.

if $f(a) = a$ or $f(b) = b$ then we are done.

So assume that $f(a) \neq a$ $(f(a) > a)$
and $f(b) \neq b$ $(f(b) < b)$

So $g(a) = f(a) - a > 0$, and
$g(b) = f(b) - b < 0$

thus $\exists c \in (a,b)$ s.t $g(c) = 0$ (Intermediate Value theorem)

$g(c) = f(c) - c = 0$
$\Rightarrow f(c) = c$
c is thus a fixed point of f.

Remark (The intermediate Value Theorm)

Let $f: I \to I$ be a continuous map. Then, for any real number r between $f(a)$ and $f(b)$, there exists $c \in I$ such that $f(c) = r$.

Theorem

Let $f: I = [a,b] \to \mathbb{R}$ be a continuous map such that $f(I) \supset I$. Then f has a fixed point in I.

Remark

Even if fixed points of a map do exist, it is sometimes not possible to compute them algebraically. For example, to find the fixed points of the map $f(x) = 2\sin x$, one needs to solve the transcendental equation $2\sin x - x = 0$.

Clearly $x = 0$ is a root of this equation and thus a fixed point of the map f.

However, the other two fixed points may be found by graphical or numerical methods. They are approximately $\pm$ 1.944795452.

Graphical Iteration and Stability

One of the main objectives in the theory of dynamical systems is the study of the behavior of orbits near fixed points, i.e the behavior of solutions of a difference equation near equilibrium points. Such a program of investigation is called stability theory.

Definition:
Let $f: I \longrightarrow I$ be a map and p be a fixed point of f, where I is an interval in the set of real numbers R. then

1- p is said to be stable if for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_0 \in I$, if $|x_0 - p| < \delta$, then $|f^n(x_0) - p| < \epsilon$ for all $n \in Z^+$

Otherwise, the fixed point p will be called unstable.

2- p is said to be attracting if there exists $\delta > 0$ such that for all $x_0 \in I$, if $|x_0 - p| < \delta$, then $\lim_{n \to \infty} f^n(x_0) = p$

3- p is asymptotically stable if it is both stable and attracting.

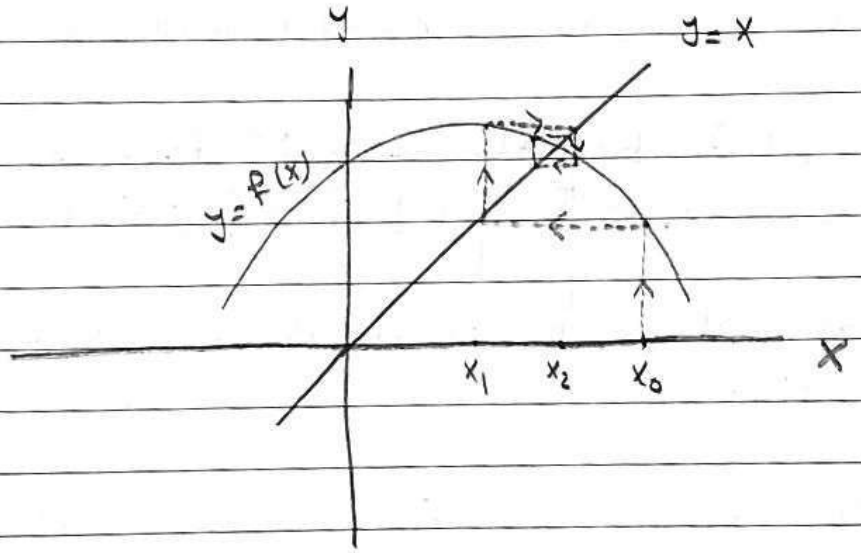
If in (2) $\delta = \infty$, then p is said to be globally asymptotically stable.

Remark: Let Z^+ denote the set of nonnegative integers.

The Cobweb Diagram (Stair-step diagram)

One of the most effective graphical iteration methods to determine the stability of fixed points is the cobweb diagram. On the x-y plane, we draw the curve $y = f(x)$ and the diagonal line $y = x$ on the same plot. We start at an initial point x_0. Then we move vertically until we hit the graph of f at the point $(x_0, f(x_0))$. We then travel horizontally to meet the line $y = x$ at the point $(f(x_0), f(x_0))$. This determines $f(x_0)$ on the x axis. To find $f^2(x_0)$, we move again vertically until we strike the graph of f at the point $(f(x_0), f^2(x_0))$ and then we move horizontally to meet the line $y = x$ at the point $(f^2(x_0), f^2(x_0))$. Continuing this process we can evalute all of the points in the orbit of x_0 namely the set

$$\{x_0, f(x_0), f^2(x_0), \ldots, f^n(x_0), \ldots\}$$



Criteria for Stability

Fixed (equilibrium) points may be divided into two types hyperbolic and nonhyperbolic.

A fixed point p of a map f is said to be hyperbolic if $|f'(p)| \neq 1$

p is said to be nonhyperbolic if $|f'(p)| = 1$

Theorem 1.
Let p be a hyperbolic fixed point of a map f, where f is continuously differentiable at p. The following statements then hold true:

1- If $|f'(p)| < 1$, then p is asymptotically stable.

2- If $|f'(p)| > 1$, then p is unstable.

Example
Consider the Logistic map $L_M(x) = Mx(1-x)$ defined on the interval $[0,1]$, where $M \in (0,4]$. Find the fixed points of $L_M(x)$ and determine their Stability.

Sol. To find the fixed points

$$L_M(x) = x \Rightarrow Mx(1-x) = x$$
$$\therefore Mx^2 - Mx + x = 0$$
$$\text{So } x = 0 = p_0 \quad \text{or} \quad x = \frac{M-1}{M} = 1 - \frac{1}{M} = p_1$$

$L'_M(X) = M - 2MX$

For $p_0 = 0$, $|L'(p_0)| = M$

So if $M < 1$ then p_0 is asymptotically stable.
and So if $M > 1$ then p_0 is unstable (theorem 1)

For $p_1 = 1 - \frac{1}{M}$, $|L'_M(p_1)| = |M - 2M(1 - \frac{1}{M})|$

$= |2 - M|$

So if $|L'_M(p_1)| = |2 - M| < 1$ then

p_1 is asymptotically stable

$|2 - M| < 1 \longrightarrow -1 < 2 - M < 1 \rightarrow 1 < M < 3$,

and if $|2 - M| > 1$, then p_1 is unstable

$2 - M > 1$ or $2 - M < -1$

$M < 1$ or $M > 3$

So if $0 < M < 1$ or $3 < M \leq 4$ then

p_1 is unstable.

Theorem 2

Let p be a fixed point of a map f such that $f'(p)=1$. If $f'(x)$, $f''(x)$, and $f'''(x)$ are continuous at p, then the following statements hold:

1- If $f''(p) \neq 0$, then p is semistable.

2- If $f''(p)=0$ and $f'''(p)>0$, then p is unstable.

3- If $f''(p)=0$ and $f'''(p)<0$, then p is asymptotically stable.

Example Let $f(x)=-x^3+x$

Find the fixed point and determine their Stability.

Sol $-x^3+x=x \rightarrow x=0=p_0$

$f'(x)=-3x^2+1 \rightarrow f'(0)=1$ So p_0 is nonhyperbolic

$f''(x)=-6x$, $f''(0)=0$,

$f'''(x)=-6$, $f'''(0)=-6<0$

Thus $p_0=0$ is asymptotically stable (by th.2)

Definition:- The Schwarzian derivative, Sf, of a function f is defined by

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2.$$

And if $f'(p) = -1$, then

$$Sf(p) = -f'''(p) - \frac{3}{2}[f''(p)]^2$$

Theorem-3-

Let p be a fixed point of a map f such that $f'(p) = -1$. If $f'(x)$, $f''(x)$ and $f'''(x)$ are continuous at p, then the following statements hold:

1- If $Sf(p) < 0$, then p is asymptotically stable.
2- If $Sf(p) > 0$, then p is unstable.

Example:- Consider the map $f(x) = x^2 + 3x$ on the interval $[-3, 3]$. Find the equilibrium points and then determine their stability.

Sol $x^2 + 3x = x \Longrightarrow x^2 + 2x = 0$

$x(x+2) = 0 \Longrightarrow x = 0 = p_0$

or $x = -2 = p_1$

For stability of p_0

$f'(x) = 2x + 3$, $f''(x) = 2$, $f'''(x) = 0$

$|f'(0)| = 3 > 1 \Longrightarrow p_0$ is repelling (unstable by the 1)

For $p_1 = -2$, $f'(-2) = -1$, $f''(-2) = 2$

$Sf(-2) = 0 - \frac{3}{2}(2)^2 = -6$

Thus $Sf(-2) < 0$, $p_1 = -2$ is (asymptotically stable) (attracting)

Periodic Points and their Stability

The notion of periodicity is one of the most important notion in the field of dynamical systems.
Its importance stems from the fact that many physical phenomena have certain patterns that repeat themselves. These patterns produce cycles (or periodic cycles), where a cycle is understood to be the orbit of a periodic point.

Definition
Let $\bar{x}$ be in the domain of a map f. Then.

1- $\bar{x}$ is said to be a periodic point of f with period k if $f^k(\bar{x}) = \bar{x}$ for some positive integer k. In this case $\bar{x}$ may be called k-periodic.

If in addition $f^r(\bar{x}) \neq \bar{x}$ for $0 < r < k$, then k is called the minimal period of $\bar{x}$.

Note that $\bar{x}$ is k-periodic if it is a fixed point of the map f^k.

2- $\bar{x}$ is said to be an eventually periodic point of a period k and delay m if
$f^{k+m}(\bar{x}) = f^m(\bar{x})$ for some positive integer k and $m \in \mathbb{Z}^+$

if $k=1$, then $f(f^m(\bar{x})) = f^m(\bar{x})$ and $\bar{x}$ is then an eventually fixed point, and if $m=0$, then $\bar{x}$ is k-periodic.

Example:- If $f(x) = |x|$, $x \in R$.

Find Fix(f) , EFix(f) , periodic points

Sol

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

For fix. pts ① $x = x \rightarrow \forall x \geq 0$

② $-x = x \rightarrow x = 0$

$\therefore [0, \infty)$ all are fixed points of f.

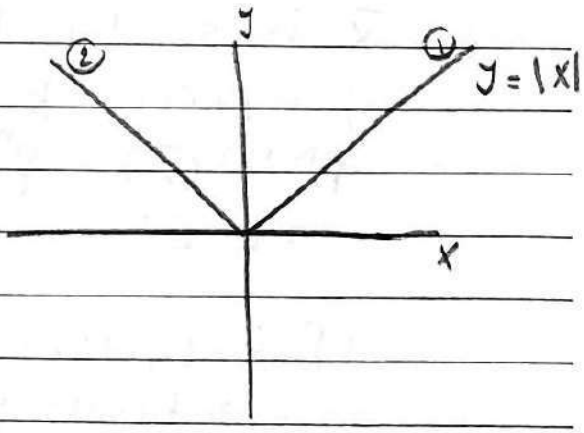
if $x < 0 \rightarrow |x| > 0$ (if $x < 0 \rightarrow f(x) = -x > 0 \rightarrow x \in EFix(f)$

$\therefore$ x is an eventually fix. pt

So $Fix(f) = [0, \infty)$,

$EFix(f) = R$,

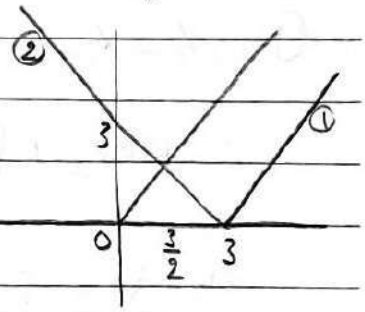
periodic pts. of per. $k \geq 2 = \phi$



Example:- If $f(x)=|x-3|$, $x\in R$.
Find all fixed points , and all periodic-2 point

Sol

$$f(x)=|x-3|=\begin{cases} x-3 , & x-3\geq 0 \quad (x\geq 3) \\ 3-x , & x-3<0 \quad (x<3) \end{cases}$$



For fixed point ($f(x)=x$)

① $x-3=x$ has no solution

② $3-x=x \longrightarrow x=\frac{3}{2}$ the only Fix.pt of f.

So $\text{Fix}(f)=\{\frac{3}{2}\}$

To find periodic-2 we must solve $f^2(x)=x$

$$f^2(x)=f(f(x))=f\left(\begin{cases} x-3 , & x\geq 3 \\ 3-x , & x<3 \end{cases}\right)$$

$$=\begin{cases} x-3-3 , & x\geq 3 , \ x-3\geq 3 \implies x\geq 6 \\ 3-(x-3) , & x\geq 3 , \ x-3<3 \implies x\in[3,6) \\ 3-x-3 , & x<3 , \ 3-x\geq 3 \implies x\leq 0 \\ 3-(3-x) , & x<3 , \ 3-x<3 \implies x\in(0,3) \end{cases}$$

So

$$f^2(x)=\begin{cases} x-6 , & x\geq 6 \\ 6-x , & 3\leq x<6 \\ -x , & x\leq 0 \\ x , & 0<x<3 \end{cases}$$

① $x-6=x$ has no solution

② $6-x=x \Rightarrow x=3$

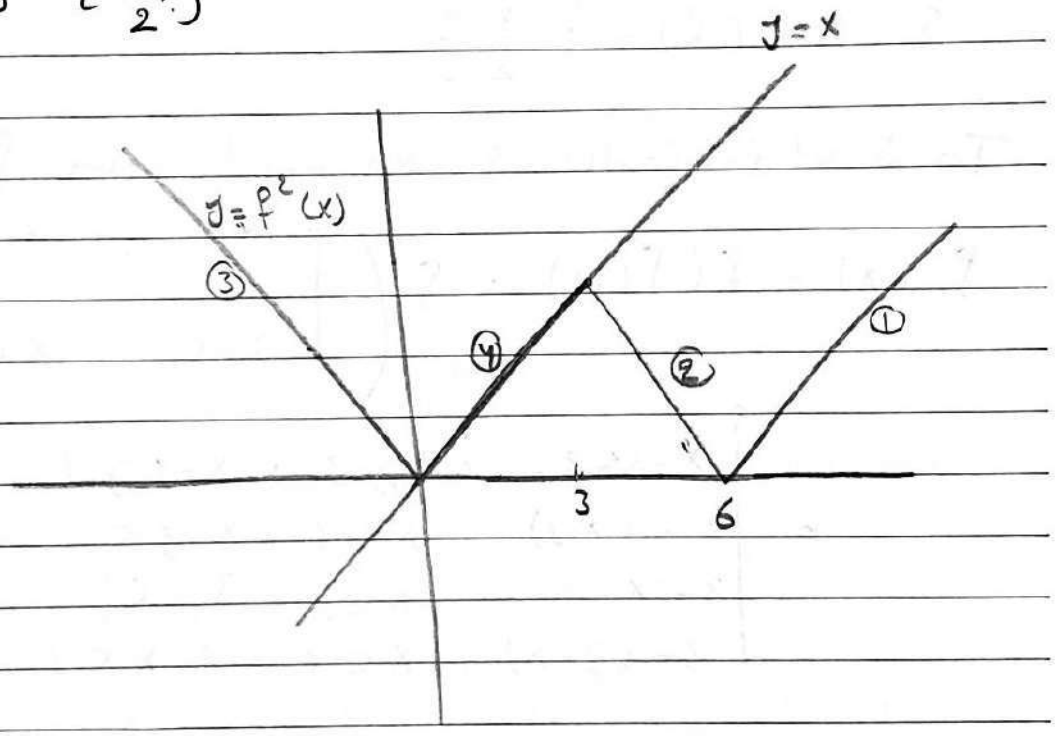
③ $-x=x \Rightarrow x=0$

④ $x=x$, $0<x<3$

$\therefore Per_2(f) = [0,3]$

all the perodic-2 of f are in the interval

$[0,3]-\{\frac{3}{2}\}$



Definition:

Let $\bar{x}$ be a periodic point of f with minimal period k. then

1- $\bar{x}$ is stable if it is a stable fixed point of f^k.

2- $\bar{x}$ is asymptotically stable if it is an asymptotically stable fixed point of f^k.

3- $\bar{x}$ is unstable if it is an unstable fixed point of f^k.

Theorem:

Let $O(\bar{x}) = \{\bar{x}, f(\bar{x}), \ldots, f^{k-1}(\bar{x})\}$ be the orbit of the k-periodic point $\bar{x}$, where f is a continuously differentiable function at $\bar{x}$. Then the following statements hold true.

1- $\bar{x}$ is asymptotically stable if

$$|f'(\bar{x})\, f'(f(\bar{x})) \cdots f'(f^{k-1}(\bar{x}))| < 1$$

2- $\bar{x}$ is unstable if

$$|f'(\bar{x})\, f'(f(\bar{x})) \cdots f'(f^{k-1}(\bar{x}))| > 1$$

Example The tent map T is defined as

$$T(x)=\begin{cases}2x & 0\le x\le \frac{1}{2}\\ 2(1-x) & \frac{1}{2}< x\le 1\end{cases}$$

Find the 2-periodic cycles of T and determine their stability.

Solution First we find the fixed points of T

$2x=x \Rightarrow x=0$ fixed point of T

$2(1-x)=x \Rightarrow x=\frac{2}{3}$ fixed point of T

$Fix(T)=\{0,\frac{2}{3}\}$

Now periodic-2 we must solve $T^2(x)=x$

$$T^2(x)=T(T(x))=T\left(\begin{cases}2x, & 0\le x\le \frac{1}{2}\\ 2(1-x), & \frac{1}{2}<x\le 1\end{cases}\right)$$

$$T^2(x)=\begin{cases}2(2x), & 0\le x\le \frac{1}{2}, & 0<2x\le \frac{1}{2}\\ 2(1-2x), & 0\le x\le \frac{1}{2}, & \frac{1}{2}<2x\le 1\\ 2(2(1-x)), & \frac{1}{2}\le x\le 1, & 0\le 2(1-x)\le \frac{1}{2}\\ 2(1-2(1-x)), & \frac{1}{2}<x\le 1, & \frac{1}{2}<2(1-x)\le 1\end{cases}$$

$$T^2(x) = \begin{cases} 4x, & 0 \le x \le \frac{1}{4} \\ 2(1-2x), & \frac{1}{4} < x \le \frac{1}{2} \\ 4(1-x), & \frac{3}{4} \le x \le 1 \\ 2(2x-1), & \frac{1}{2} < x < \frac{3}{4} \end{cases}$$

① $4x = x \Rightarrow x = 0$ omit fix. pt. of T

② $2(1-2x) = x \Rightarrow x = \frac{2}{5}$ periodic-2 point of T
(fixed point of T^2)

③ $4(1-x) = x \Rightarrow x = \frac{4}{5}$ " "
" "

④ $2(2x-1) = x \Rightarrow x = \frac{2}{3}$ omit fix. pt. of T

∴ all periodic-2 points are $\frac{2}{5}, \frac{4}{5}$

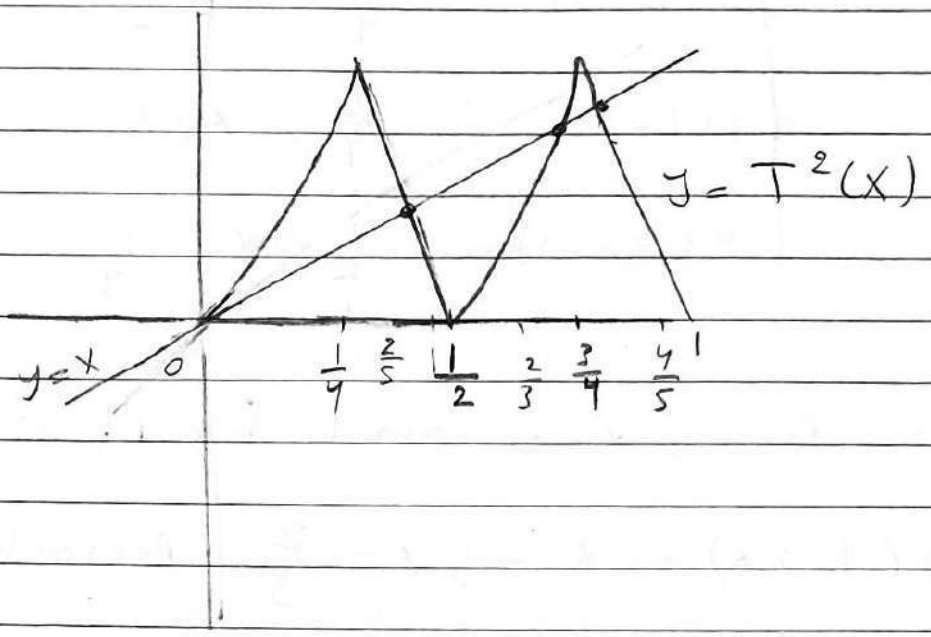
2-periodic cycle is only $\{\frac{2}{5}, \frac{4}{5}\}$

$|T'(\frac{2}{5}) \cdot T'(\frac{4}{5})| = 4 > 1$

$(|T'(x)| = 2 \quad \forall x \neq \frac{1}{2})$

∴ $\{\frac{2}{5}, \frac{4}{5}\}$ unstable

$Per_2(T) = \{0, \frac{2}{3}, \frac{2}{5}, \frac{4}{5}\}$



Remark

$f : I \xrightarrow{\text{map}} I$

k given positive integer

$$Per_k(f) = \{x : f^k(x) = x\}$$

$$per(f) = \{x : f^k(x) = x, \ \forall k \in \mathbb{Z}^+\}$$

$$Per_k(f) \subseteq Per(f)$$

The period-Doubling Route to Chaos

We end this chapter by studying in detail the logistic map.

$$L_M(x) = Mx(1-x), \quad x \in [0,1] \text{ and } M \in (0,4]$$

which gives rise to the logistic difference equation

$$x(n+1) = M x(n)(1-x(n))$$

Fixed Point x

$$Mx(1-x) = x \Rightarrow Mx - Mx^2 = x \Rightarrow Mx^2 - Mx + x = 0$$

$$x(Mx - M + 1) = 0 \rightarrow x = 0 = p_0,$$

$$Mx - M + 1 = 0 \Rightarrow x = \frac{M-1}{M} = 1 - \frac{1}{M} = p_1$$

We now examine the stability of each fixed separately.

$$L'_M(x) = M - 2Mx$$

For p_0, $L'_M(0) = M$, So if $M < 1$ then p_0 is attracting

p_0 is asymptotically stable if $0 < M < 1$

if $M > 1$ then p_0 is unstable (repelling)

thus $M = 1$ is a bifurcation point.

For $p_1 = 1 - \frac{1}{M}$, $|L'_M(p_1)| = |M - 2M(1 - \frac{1}{M})| = |2 - M|$

So, if $|2-M| < 1$ then p_1 is attracting.

$-1 < 2 - M < 1 \Rightarrow 1 < M < 3$ (asymptotically stable)

if $|2-M| > 1$ then p_1 is unstable (repelling)

$2-M > 1$ or $2-M < -1 \Rightarrow M < 1$ or $M > 3$

Thus $M = 3$ is a bifurcation point

In general L_M has infinite bifurcation points.

To find the 2-periodic cycles we solve the eq.

$$L_M^2(x) = x \quad , \quad L_M(L_M(x)) = x$$

$$L_M(MX(1-X)) = X$$

$$M^2 X(1-X)[1 - MX(1-X)] - X = 0$$

Discarding the equilibrium points 0 and $\frac{M-1}{M}$ by dividing the left side by $x(x - \frac{M-1}{M})$, we obtain

$$M^2 x^2 - M(M+1)x + (M+1) = 0$$

Solving this equation.

$$\bar{x}_1 = \frac{(1+M) - \sqrt{(M-3)(M+1)}}{2M} \quad \text{and}$$

$$\bar{x}_2 = \frac{(1+M) + \sqrt{(M-3)(M+1)}}{2M}$$

Clearly $\bar{x}_1$ and $\bar{x}_2$ are defined only if $M > 3$.

Next, we investigate the stability of this 2-Periodic cycle.

This 2-periodic cycle is asymptotically stable if

$$\left| L'_M(\bar{x}_1)\, L'_M(\bar{x}_2) \right| < 1$$

or

$$-1 < M^2 (1 - 2\bar{x}_1)(1 - 2\bar{x}_2) < 1$$

$$-1 < M^2 \left(1 - \frac{(1+M) - \sqrt{(M^2-2M-3)}}{M}\right)\left(1 - \frac{(1+M) + \sqrt{(M^2-2M-3)}}{M}\right) <$$

$$-1 < -M^2 + 2M + 4 < 1$$

Solving the last two inequalities yields the range:

$3 < M < 1 + \sqrt{6}$ for asymptotic stability.

Now, for $M = 1 + \sqrt{6}$,

$$L'_M(\bar{x}_1)\, L'_M(\bar{x}_2) = -1$$

After some computation, we conclude that $S L^2_M(\bar{x}_1) < 0$ and $S L^2_M(\bar{x}_2) < 0$ which implies that the cycle $\{\bar{x}_1, \bar{x}_2\}$ is asymptotically stable. Moreover, the periodic cycle $\{\bar{x}_1, \bar{x}_2\}$ is unstable for $M > 1 + \sqrt{6}$

Thus, the positive equilibrium point is asymptotically stable for $1 < M < 3$, where it loses its stability after $M_1 = 3$. For $M > M_1$, an asymptotically stable 2-periodic cycle appears where it loses its stability after a second magic number $M_2 = 1 + \sqrt{6}$, etc.

2^2-periodic Cycles

The search for 4-periodic cycles can be successful if one is able to solve the equation $L_\mu^4(x) = x$. This involves solving a twelfth-degree equation, which is not possible in general. So we turn to graphical or numerical analysis to help us find the 4-periodic cycles.

The best way to illustrate this discussion is to draw the so-called bifurcation diagram.

We let the horizontal axis represent the parameter μ and the vertical axis representing higher iterates $L_\mu^n(x_0)$ of a specific initial point x_0, so the diagram will show the limiting behavior of the orbit of x_0.

"Chapter two" "Attraction and Bifurcation"

Basin of Attraction

Let p be a fixed point of map f. Then the basin of attraction (or the stable set) $W^s(p)$ of p is defined as

$$W^s(p) = \{ x : \lim_{n \to \infty} f^n(x) = p \}$$

In other words, $W^s(p)$ consists of all points that are forward asymptotic to p.

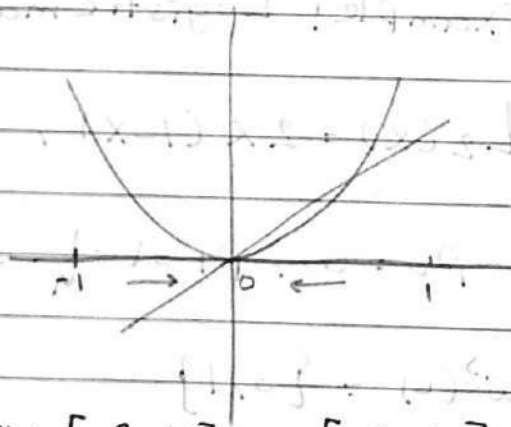
The maximal interval in $W^s(p)$ that contains p is called the immediate basin of attraction and is denoted by $B(p)$

Example: The map $f(x) = x^2$, $x \in R$
Find all fixed points and their basin of attraction

Sol $x = 0 = p_0$, $x = 1 = p_1$

$W^s(0) = (-1, 1)$

$W^s(1) = \{-1, 1\}$



Example: Consider the map $g : [-2, 4] \to [-2, 4]$ defined as

$$g(x) = \begin{cases} x^2 & \text{if } -2 \le x \le 1 \\ 3\sqrt{x} - 2 & \text{if } 1 \le x \le 4 \end{cases}$$

Find all fixed points and their basin of attraction

Sol $x^2 = x \rightarrow x = 0 = p_0$, $x = 1 = p_1$

$3\sqrt{x} - 2 = x \rightarrow 3\sqrt{x} = x + 2 \rightarrow 9x = x^2 + 4x + 4$

$\therefore x^2 - 5x + 4 = 0 \rightarrow (x-4)(x-1) = 0$

$\therefore x = 4 = p_2$

$W^s(0) = (-1, 1)$

$W^s(1) = \{-1, 1\}$

$W^s(4) = [-2, -1) \cup (1, 4]$

$B(4) = (1, 4]$

$B(0) = (-1, 1)$

$B(1) = \emptyset$

Example: Logistic map ($M = 2$)

$L_2(x) = 2x(1-x)$, $x \in [0, 1]$

$p_0 = 0$, $p_1 = 1 - \frac{1}{M} = \frac{1}{2}$

$W^s(0) = \{0, 1\}$

$W^s(\frac{1}{2}) = (0, 1) = B(\frac{1}{2})$

Remark: The basin of attraction of the two fixed points are disjoint.

This due to the uniqueness of the limit of a sequence

In other words,

if $\lim_{n\to\infty} f^n(x) = L_1$ and $\lim_{n\to\infty} f^n(x) = L_2$ then $L_1 = L_2$

Definition: The set $M \subseteq \text{Dom } f$ is said to be an invariant set if $f(M) \subseteq M$.

i.e $O(x) \subseteq M$, $\forall x \in M$

Theorem:

Let $f: I \to \mathbb{R}$ be a continuous map from an interval I to $\mathbb{R}$ and let p be an asymptotically stable fixed point of f. Then $W^s(p)$ is a nonempty open and (positively) invariant set.

Definition: A continuous map $f: [0,\infty) \to [0,\infty)$ is said to be a population model if

① $f(0) = 0$, and f has a unique positive fixed point p

② if $0 < x < p$, then $f(x) > x$
if $x > p$, then $f(x) < x$

③ If $f'(x_m) = 0$ and $x_m \leq p$ then $f'(x) > 0$ for $0 \leq x \leq x_m$ and $f'(x) \leq 0$ for $x > x_m$ and $f(x) > 0$.

Theorem: The positive fixed point of a population model is globally asymptotically stable on $(0,\infty)$ if and only if it has no periodic 2 cycle.

Idea

Singer's Theorem

Recall that the Schwarzian derivative Sf of a map f is defined by the formula

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

It turns out that many polynomials have negative Schwarzian derivatives.

Lemma: Let $p(x)$ be a polynomial of degree n, $p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$, such that all the roots of its derivative $p'(x)$ are distinct and real. Then $Sp(x) < 0$ ($-\infty$ is allowed).

Proof Let $r_1, r_2, \ldots, r_{n-1}$ be the real and distinct roots of $p'(x)$. Then

$$p'(x) = a(x-r_1)(x-r_2)\cdots(x-r_{n-1}), \text{ where } a = na_0$$

Taking the natural logarithm of both sides

$$\ln p'(x) = \ln a + \sum_{i=1}^{n-1} \ln(x-r_i)$$

Differentiating with respect to x we obtain

$$\frac{p''(x)}{p'(x)} = \sum_{i=1}^{n-1} \frac{1}{(x-r_i)} \quad \cdots\cdots (*)$$

Differentiating one more time we get

$$\frac{p'(x)\,p'''(x) - (p''(x))^2}{(p'(x))^2} = \frac{p'''(x)}{p'(x)} - \left(\frac{p''(x)}{p'(x)}\right)^2$$

$$= -\sum_{i=1}^{n-1} \frac{1}{(x-r_i)^2} \qquad \text{..... (**)}$$

Now using (*) and (**) yields

$$-\frac{1}{2}(*)^2 + (**)$$

$$\frac{p'''(x)}{p'(x)} - \left(\frac{p''(x)}{p'(x)}\right)^2 - \frac{1}{2}\left(\frac{p''(x)}{p'(x)}\right)^2 = -\sum_{i=1}^{n-1} \frac{1}{(x-r_i)^2} - \frac{1}{2}\left(\sum_{i=1}^{n-1} \frac{1}{x-r_i}\right)^2$$

$$\therefore Sp(x) < 0$$

Lemma:

Assume that f is a C^3 map on $\mathbb{R}$ and $Sf(x) < 0$. Then the following statements hold true.

(i) If u is a point at which $f'(x)$ has a local minimum, then $f'(u) \leq 0$, and if v is a point at which $f'(x)$ has a local maximum, then $f'(v) \geq 0$.

In other words, $f'(x)$ cannot have a positive local minimum or a negative local maximum.

(ii) Let a_1, a_2, a_3 be three fixed points of f such that $a_1 < a_2 < a_3$. If $f'(a_2) \leq 1$, then f has a critical point in the interval (a_1, a_3), i.e there exists a point $d \in (a_1, a_3)$ with $f'(d) = 0$.

We need one more technical result before stating Singer's Theorem: the product rule for the Schwarzian derivative.

Product Rule:

$$\boxed{S(f \circ g)(x) = Sf(g(x)) \cdot (g'(x))^2 + Sg(x)}$$

Corollary:

(i) If $Sf < 0$ and $Sg \leq 0$, then $S(f \circ g) < 0$.

(ii) If $Sf < 0$, then $Sf^k < 0$ for all $k \in \mathbb{Z}^+$.

Theorem:

Suppose that f is a C^3 map defined on the closed interval $I = [a, b]$, $a = -\infty$, $b = \infty$ are allowed, such that $Sf(x) < 0$ for all $x \in I$ ($Sf(x) = -\infty$ is allowed).

If p is an attracting k-periodic point, then either

(i) the immediate basin of attraction $W^s(p)$ extends to a or to b ($-\infty$ or $+\infty$), or

(ii) the immediate basin of attraction $W^s(p)$ contains a critical point x_c ($f'(x_c) = 0$).

Corollary (Singer, 1978):

Suppose that f is a C^3 map on a closed interval I such that $Sf(x) < 0$, for all $x \in I$. If f has n critical points in I, then for every $k \in \mathbb{Z}^+$, f has at most $(n+2)$ attracting k-periodic orbits.

Idea Plus

Example: Show that the map $f(x) = 1 - 2x^2$ on $[-1,1]$ has no attracting periodic points.

Sol $Sf < 0$

$f(x) = 1 - 2x^2 \longrightarrow f'(x) = -4x = 0$

$\therefore x = 0$ is the only critical point of f

$O(0) = \{0, 1, -1\} \longrightarrow -1$ Thus -1 is a possible attracting fixed point of f.

But, this impossible because $|f'(-1)| = 4 > 1 \longrightarrow x = -1$ unstable

$\therefore f$ has no attracting periodic point.

Example:- Consider the map $f(x) = \frac{1}{3}x^3 + 2x$, Show that $Sf(x) \geq 0$, $\forall x \in [-1,1]$.

Sol $Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2$

$f'(x) = x^2 + 2$, $f''(x) = 2x$, $f'''(x) = 2$

$Sf(x) = \frac{2}{x^2+2} - \frac{3}{2}\left[\frac{2x}{x^2+2}\right]^2$

$= \frac{2(2(x^2+2)) - 3(4x^2)}{2(x^2+2)^2} \geq 0$

$\longrightarrow 4(x^2+2) - 12x^2 \geq 0$

$\longrightarrow -1 \leq x \leq 1$

Example: The logistic map $L_M(x) = Mx(1-x)$
$x \in [0,1]$, $0 < M \leq 4$

$S L_M(x) < 0$

$L'_M(x) = M - 2Mx = 0 \longrightarrow x = \frac{1}{2}$ (has only one critical point) $(n=1)$

$p_0 = 0$, $p_1 = 1 - \frac{1}{M}$ (fixed points) , $L_M(1) = 0$

As we saw in chapter 1, for $0 < M \leq 1$
0 is the only attracting fixed point,
where the region of attraction is $[0,1]$
$W^s(0) = [0,1]$

For $1 \leq M < 4$ (L_M has only one critical point $\frac{1}{2}$)
There are at most three attracting periodic cycles
since $L'_M(0) = M > 1$, the fixed point 0 is unstable.
Furthermore $L_M(1) = 0$
We conclude that there is at most one attracting periodic cycle for given $M \in (1,4]$.

IF $M = 4$, then $L_4(x) = 4x(1-x)$
$p_0 = 0$ unstable , $p_1 = \frac{3}{4}$

$L'_4(x) = 4 - 8x \longrightarrow |L'_4(\frac{3}{4})| = |-2| = 2 > 1$
$\therefore p_1$ unstable

$O(\frac{1}{2}) = \{\frac{1}{2}, 1, 0\} \longrightarrow 0$
So L_4 hasn't attracting periodic cycle with cycle k
$\forall k \in Z^+$
idea L_4 chaotic map

Example: Let $G(x) = M \tan^{-1} x$, $M \neq 0$

then $SG(x) = \frac{-2}{(x^2+1)^2} < 0$

$G'(x) = \frac{M}{1+x^2} \neq 0$

Clearly $G(x)$ has no critical points

Now, if $|M| < 1$, then $p_0 = 0$ is an asymptotically stable fixed point
basin of attraction $W^s(p_0) = (-\infty, \infty) = \mathbb{R}$

while if $M > 1$, then G has two attracting fixed points p_1 and p_2 with basins of attraction

$W^s(p_1) = (-\infty, 0)$, $W^s(p_2) = (0, \infty)$

Finally, if $M < -1$, then G has an attracting 2-cycle $\{\bar{p}_1, \bar{p}_2\}$
with basin of attraction of the form

$W^s(\{\bar{p}_1, \bar{p}_2\}) = (-\infty, 0) \cup (0, \infty)$

Bifurcation

Types of bifurcation of fixed points

Bifurcation	$\frac{\partial H}{\partial x}(M^*, x^*)$	$\frac{\partial H}{\partial M}(M^*, x^*)$	$\frac{\partial^2 H}{\partial x^2}(M^*, x^*)$
saddle-node	1	$\neq 0$	$\neq 0$
pitchfork	1	0	0
Transcritical	1	0	$\neq 0$
period-doubling (flip)	-1	$\neq 0$	$\neq 0$

proof of Singer theorem

If there are n critical points then by theorem (ii)

(Suppose that f is a C^3 map defined on the closed interval $I = [a,b]$, $a = -\infty$, $b = \infty$ are allowed. Such that $Sf(x) < 0$ for all $x \in I$ ($Sf(x) = -\infty$ is allowed). If p is an attracting k-periodic point, then either

(ii) the immediate basin of attraction $W^s(p)$ contains a critical point x_c ($f'(x_c) = 0$)

there are n attracting k-periodic cycles for any given k.

We may have two more additional attracting k-periodic cycle by theorem (i)

(i) the immediate basin of attraction $W^s(p)$ extend to a or to b ($-\infty$ or $+\infty$))

Thus the maximum number of attracting periodic cycles is $n+2$

Schwarzian Min-Max principle

Assume that f is a C^3 map on R and $Sf(x) < 0$ then $f'(x)$ can not have a positive local minimum or a negative local maximum.

Proof

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2$$

if x_0 is a critical point for f'

then $f''(x_0) = 0$

$$\rightarrow Sf(x_0) = \frac{f'''(x_0)}{f'(x_0)}$$

if x_0 is a positive local minimum for f', then

$f'''(x_0) \geq 0$ (by the second derivative test)

$f'(x_0) > 0$ (Since we assume x_0 is a positive local min for f')

$\rightarrow Sf(x_0) \geq 0$

this contradicts $Sf < 0$

Similarly, if x_0 is a negative local maximum, then

$f'''(x_0) \leq 0$ (by the second derivative test)

$f'(x_0) < 0$ (Since we assume x_0 is a negative local max for f')

thus $Sf(x_0) \geq 0$ this a contraduction.

Sharkovsky's Theorem

First, let us turn our attention to Sharkovsky's ordering of the positive integers. This ordering is defined as follows:

$3 \rhd 5 \rhd 7 \rhd \cdots$ odd integers

$2\times 3 \rhd 2\times 5 \rhd 2\times 7 \rhd \cdots$ $2 \times$ odd integers

$2^2\times 3 \rhd 2^2\times 5 \rhd 2^2\times 7 \rhd \cdots$ $2^2 \times$ odd integers

$2^n\times 3 \rhd 2^n\times 5 \rhd 2^n\times 7 \rhd \cdots$ $2^n \times$ odd integers

$\cdots \rhd 2^n \rhd 2^{n-1} \rhd \cdots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1$ powers of 2

We first list all the odd integers except 1, then 2 times the odd integers, 2^2 times the odd integers, and, in general, 2^n times the odd integers for all $n \in \mathbb{Z}^+$. This is followed by powers of 2 in a descending order. It is easy to see that this ordering exhausts all of the positive integers.

Theorem:
Let $f: I \to I$ be a continuous map on the interval I, where I may be finite, infinite, or the whole real line.

If f has a periodic point of period k, then it has a periodic point of period r for all r with $k \rhd r$.

Theorem (Li and Yorke)

Let $f: I \to I$ be a continuous map on an interval I. If there is a periodic point in I of period 3, then for every $k = 1, 2, \ldots$ there is a periodic point in I having period k.

Example:

$$f(x) = \begin{cases} x + \frac{1}{2} & \text{for } 0 \le x < \frac{1}{2}, \\ 2 - 2x & \text{for } \frac{1}{2} \le x \le 1. \end{cases}$$

The map f has the periodic orbit $\{0, \frac{1}{2}, 1\}$ is 3-cycle. Moreover, f has the fixed point $\frac{2}{3}$.

Let us write every $x \in [0,1]$ in its binary expansion.

$$x = \sum_{j=1}^{\infty} \frac{a_j}{2^j} = 0.a_1 a_2 a_3 \ldots,$$

where each a_j is either 0 or 1.
Now for $0 \le x < \frac{1}{2}$, $a_1 = 0$ and thus

$$\begin{aligned} f(x) &= \frac{1}{2} + 0.0 a_2 a_3 \\ &= (0.1\bar{0})_2 + (0.0 a_2 a_3 \ldots)_2 \\ &= 0.1 a_2 a_3 \ldots \end{aligned}$$

On the other hand if $\frac{1}{2} \le x \le 1$, $a_1 = 1$ and hence

$$\begin{aligned} f(x) &= 2 - 2(0.1 a_2 a_3 \ldots) = 2 - 1.a_2 a_3 a_4 \ldots \\ &= 1 - 0.a_2 a_3 a_4 \ldots \\ &= (0.1\bar{1}) - 0.a_2 a_3 a_4 \ldots \\ &= 0.a_2' a_3' a_4' \ldots \end{aligned}$$

$f(x) = 0.a_2' a_3' a_4 \cdots$ where $a_j' = 1 - a_j$.

One may show that

$$f^2(0.1_0 a_3 a_4 \cdots) = f(0.1 a_3' a_4' \cdots)$$
$$= (0.a_3 a_4 \cdots)$$

and

$$f^2(0.0 a_2 a_3 \cdots) = f(0.1 a_2 a_3 \cdots)$$
$$= (0.a_2' a_3' \cdots)$$

To find a point of period 2 we use above equation which implies that

$a_2' = 1$, $a_3' = a_2$, $a_4' = a_3 \cdots$

Thus the point $0.0\bar{1}$ is a point of period 2

$$f(0.0\bar{1}) = f(0.0\,1\bar{0}\bar{1})$$
$$= (0.11\bar{0}\bar{1})_2$$

$\therefore \{0.0\bar{1}, 0.11\bar{0}\bar{1}\}$ is 2-cycle.

Example:- Recall that the tent map

$$T(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2(1-x), & \frac{1}{2} < x \le 1 \end{cases}$$

has two cycles of period 3.

$\{\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\}$ and $\{\frac{2}{9}, \frac{4}{9}, \frac{8}{9}\}$

So, the tent map has cycles of all periods.

A Converse of Sharkovsky's Theorem

The question that we are going to address is the following:

given any positive integers k and r with $k \triangleright r$, is there a continuous map that has a point of period r but no points of period k? The answer to this question is a definite yes.

Theorem:
For any positive integer r, there exists a continuous map $f_r : I_r \longrightarrow I_r$ on the closed interval I_r such that f_r has a point of prime period r but no points of prime periods s, for all positive integers s that precede r in the Sharkovsky's ordering, i.e

$s \triangleright \cdots \triangleright r$.

Example:
Define a continuous map $f : [1,5] \longrightarrow [1,5]$ as follows:

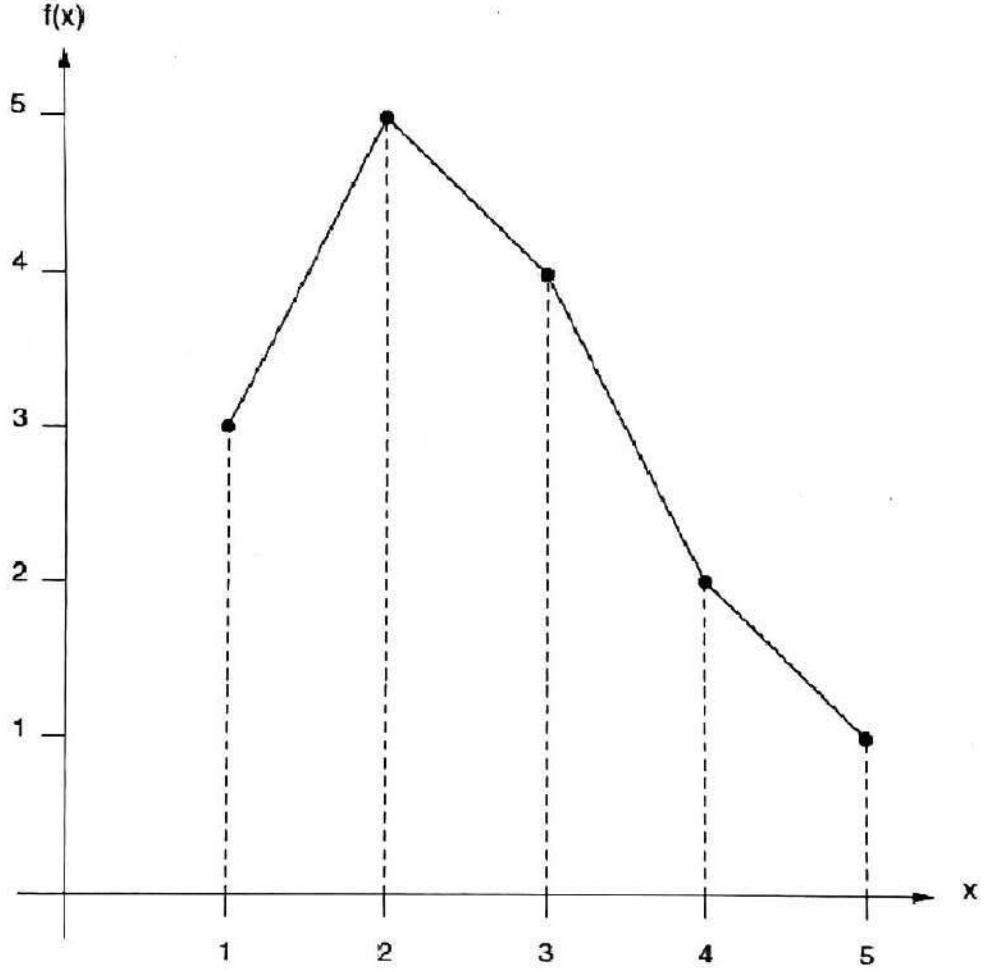
$f(1) = 3$, $f(2) = 5$, $f(3) = 4$, $f(4) = 2$ and $f(5) = 1$

f has a periodic points of period 5. Show that f hasn't a periodic point of period 3.

Solution
f has a periodic pt. of period 5

$f^5(1) = 1$, $\{1,2,3,4,5\}$ they all belong to the single 5-cycle: $1 \xrightarrow{f} 3 \xrightarrow{f} 4 \xrightarrow{f} 2 \xrightarrow{f} 5 \xrightarrow{f} 1$

Examplep. 100



A map of period 5 but no points of period 3.

on each interval $(n, n+1)$, $1 \leq n \leq 4$, f is linear (piecewise linear)

$$f^3((1,2)) = f(f(f((1,2)))) = f(f((3,5))$$

$$= f((1,4)) = (2,3)$$

$$f^3((1,2)) \cap (1,2) = (2,3) \cap (1,2) = \phi$$

f^3 has no fixed point in the interval $(1,2)$

$$f^3((2,3)) = f(f(f((2,3)))) = f(f((4,5)))$$

$$= f((1,2)) = (3,5)$$

$$f^3((2,3)) \cap (2,3) = \phi$$

f^3 has no fixed point in the interval $(2,3)$

$$f^3((3,4)) = f(f(f((3,4)))) = f(f((2,4)))$$

$$= f((2,5)) = (1,5)$$

$$f^3((3,4)) \cap (3,4) = (1,5) \cap (3,4) \neq \phi$$

We must show now that the fixed point of f^3 is really a fixed point of f and thus is not of prime period 3.

So, $(3,4)$, $(4,2)$, $m = \frac{\Delta y}{\Delta x} = \frac{2-4}{4-3} = -2$

$y - y_1 = m(x - x_1) \rightarrow y - 4 = -2(x-3)$

$y = 10 - 2x$

$f(x) = y = 10 - 2x = x \rightarrow x = \frac{10}{3}$ fixed point of f.

$$f^3(x) = f(f(f(x))) = f(f(10-2x))$$
$$= f(10-2[10-2x])$$
$$= f(10-20+4x)$$
$$= f(4x-10)$$
$$= 10-2(4x-10)$$
$$= 10-8x+20$$
$$= 30-8x$$

$f^3(x) = x$

$30-8x = x \longrightarrow x = \frac{30}{9} = \frac{10}{3}$ = fixed point of f

Hence f has no points of prime period 3, in the interval $(3,4)$

$$f^3((4,5)) = f(f(f((4,5)))) = f(f((1,2)))$$
$$= f((3,5)) = (1,4)$$

$f^3((4,5)) \cap (4,5) = \phi$

f^3 has no fixed point in the interval $(4,5)$

Hence f has no points of period 3.

Chaos in One Dimension

Density of the Set of periodic Points

Def Let I be an interval in $\mathbb{R}$. Then a set A is said to be dense in I if for any $x \in I$ any open interval containing x must intersect A.

Example:- The set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$.

Example:- The Tent Map, which is defined as

$$T(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2} \\ 2(1-x), & \frac{1}{2} < x \leq 1 \end{cases}$$

Lemma 1

A point $b \in (0,1)$ is eventually periodic under T if and only if it is rational.

Lemma 2

Let $b = \frac{r}{s}$ be a rational number in the interval $(0,1)$. Then, b is a periodic point of the tent map T if and only if r is an even integer and s is an odd integer.

Theorem

The set of periodic points of the tent map T is dense in the closed interval $[0,1]$.

Transitivity

Def. (Transitive map)
Let f be a map on an interval I (or $\mathbb{R}$). Then f is said to be topologically transitive if for any pair of nonempty open intervals J_1 and J_2 in I there exist a positive integer k such that $f^k(J_1) \cap J_2 \neq \emptyset$

i.e $\exists\, x \in I$ s.t $O(x)$ dense in I.

The doubling map $D : [0,1] \longrightarrow [0,1]$ is defined as

$$D(x) = \begin{cases} 2x & \text{for } 0 \leq x < \frac{1}{2}, \\ 2x-1 & \text{for } \frac{1}{2} \leq x < 1, \\ 0 & \text{for } x = 1. \end{cases}$$

Equivalently, the doubling map D may be defined in the compact form

$$D(x) = 2x \pmod 1,$$

where (mod 1) stand for module 1.
This means we drop the integral part leaving only the fractional part.

Theorem: Let $D:[0,1] \longrightarrow [0,1]$ be the doubling map. Then Per_D is dense in $[0,1]$ and the map D is topologically transitive.

Proof we first show that Per_D is dense in $[0,1]$.

Let $x \in [0,1]$ be written in its binary expansion.

$$x = \sum_{j=1}^{\infty} \frac{a_j}{2^j} \quad \text{where } a_j \text{ is either } 0 \text{ or } 1.$$

$$D(x) = D\left(\sum_{j=1}^{\infty} \frac{a_j}{2^j}\right)$$

$$= 2 \sum_{j=1}^{\infty} \frac{a_j}{2^j} \pmod 1$$

$$= a_1 + \sum_{j=2}^{\infty} \frac{a_j}{2^{j-1}} \pmod 1$$

$$= \sum_{j=2}^{\infty} \frac{a_j}{2^{j-1}} \quad \text{(since } a_1 = 0 \text{ or } 1 \text{ and thus omitted under (mod 1))}$$

$$D(x) = \sum_{r=1}^{\infty} \frac{a_{r+1}}{2^r}$$

Note that the nth iterate simply shifts the expansion by n places:

$$D^n(x) = D^n\left(\sum_{j=1}^{\infty} \frac{a_j}{2^j}\right) = \sum_{r=1}^{\infty} \frac{a_{r+n}}{2^r}$$

Moreover, if a point x has an expansion that repeats every n places, $a_{j+n} = a_j$ for all j, then $D^n(x) = x$, x is periodic point of period n.

Let U_δ be any nonempty open interval in $[0,1]$.

We want to prove $Per_D \cap U_\delta \neq \phi$

Let $x \in U_\delta$ $(U_\delta \neq \phi)$, $x = \sum_{j=1}^{\infty} \frac{a_j}{2^j}$ and $\delta > 0$ s.t. $\frac{1}{2^N} < \delta$

Consider the N-periodic point $(y \in Per_D)$ $y = \sum_{j=1}^{\infty} \frac{b_j}{2^j}$ in $[0,1]$, where $b_1, b_2, \cdots, b_N$ repeats and

$b_1 = a_1$, $b_2 = a_2$ $\cdots$ $b_N = a_N$

Now $|x-y| = \left| \sum_{j=1}^{\infty} \frac{a_j}{2^j} - \sum_{j=1}^{\infty} \frac{b_j}{2^j} \right|$

$$= \left| \sum_{j=N+1}^{\infty} \frac{a_j}{2^j} - \sum_{j=N+1}^{\infty} \frac{b_j}{2^j} \right|$$

$$= \left| \sum_{j=N+1}^{\infty} \frac{a_j - b_j}{2^j} \right| \leq \sum_{j=N+1}^{\infty} \frac{1}{2^j}$$

$$\therefore |x-y| \leq \sum_{j=N+1}^{\infty} \frac{1}{2^j} = \frac{\left(\frac{1}{2}\right)^{N+1}}{1-\frac{1}{2}}$$

$$= \frac{\left(\frac{1}{2}\right)^{N+1}}{\frac{1}{2}} = \frac{1}{2^N} < \delta$$

$\therefore y \in U_\delta \rightarrow Per_D \cap U_\delta \neq \phi$

Hence Per_D is dense in $[0,1]$.

Next, we show that D is topologically transitive.

Consider the point $z = \sum_{j=1}^{\infty} \frac{c_j}{2^j}$, where c_j's appear as follo

01 , 00 01 10 11 , 000 001 010 100 011 101 110 111

1-block 2-block 3-block

This sequence consist of all the blocks of 0's and 1's of length 1 (there are only 2), of length 2 (these are four 2-blocks), etc.

To prove that $O_D(z)$ dense in $[0,1]$

$$O(z) = \{z, D(z), D^2(z), D^3(z), \ldots, D^k(z), \ldots\}$$

Let U_δ be any open interval in $[0,1]$, $x \in U_\delta$

$x = \sum_{j=1}^{\infty} \frac{a_j}{2^j}$ and let $\delta > 0$ be given.

Now the string $a_1, a_2, \ldots, a_N$ must appear as one of the N-blocks in the binary expansion of z.

$\exists k \in \mathbb{Z}^+$ such that $D^k(z) = \sum_{r=1}^{\infty} \frac{c_{r+k}}{2^r}$

with $a_1 = c_{k+1}$, $a_2 = c_{k+2}, \ldots, a_N = c_{k+N}$

Now

$$\left| D^k(z) - x \right| = \left| \sum_{j=N+1}^{\infty} \frac{a_j}{2^j} - \sum_{j=N+1}^{\infty} \frac{c_{j+k}}{2^j} \right|$$

$$= \left| \sum_{j=N+1}^{\infty} \frac{q_j - c_{j+k}}{2^j} \right|$$

$$\leq \sum_{j=N+1}^{\infty} \frac{1}{2^j} = \frac{\left(\frac{1}{2}\right)^{N+1}}{1-\frac{1}{2}} = \frac{1}{2^N}$$

$$\frac{1}{2^N} < \delta$$

$$\therefore D^k(z) \in U_\delta \rightarrow O(z) \cap U_\delta \neq \emptyset$$

Hence $O(z)$ is dense in $[0,1]$

So D is transitive map

Def $f : I \longrightarrow I$ is said to be regular system if per_f dense in I.

for example D, T

Dense Set

A dense in $X \iff \bar{A} = X$

$\iff$ for any open nonempty set $U \subseteq X$, $A \cap U \neq \emptyset$

$\iff$ for any $x \in X$, and any $\delta > 0$, $\exists\, y \in A$ s.t $|x-y| < \delta$ $(d(x,y) < \delta)$

Consider the double angle map $g: S^1 \to S^1$ on the unit circle S^1 defined as $g(\theta) = 2\theta$, where θ represent the point $e^{i\theta}$ on S^1. Show that g is topologically transitive.

$g(z) = z^2$ or $(g(\theta) = 2\theta)$

$= (re^{i\theta})^2$, $\theta = \theta + 2k\pi \quad \forall k \in Z$

$= r^2 e^{i2\theta}$, $(r=1)$

$g(z) = e^{i2\theta}$, $z = e^{i\theta}$

To find fixed points

$g(z) = z \longrightarrow (z^2 = z) * z^{-1} \to z = 1 \quad (z^{-1} \text{ exist}, z \neq 0)$

To find 2-periodic points

$g^2(z) = z \to g(g(z)) = z \to (z^4 = z) * z^{-1}$

$z^{2^2-1} = 1$

To find 3-periodic points

$g^3(z) = z \Longrightarrow z^{2^3-1} = 1$

To find n-periodic points

$g^n(z) = z \to z^{2^n-1} = 1$, $|z| = 1$

$z_k = e^{\frac{i2k\pi}{2^{n-1}}} \quad k = 0, 1, 2, \ldots, 2^{n-2}$

Theorem The double angle-map $g: S^1 \to S^1$, $g(z) = z^2$ is regular System

proof

we have that the form of n-periodic points of g is

$$z_k = e^{i\frac{2k\pi}{2^{n-1}}} \qquad k = 0, 1, 2, \ldots, 2^{n-2}$$

These 2^{n-1} roots are distinct and equally spaced around the circle S^1, a distance $\frac{2\pi}{2^{n-1}}$ which can be made arbitrary small by taking n large enough.

It follows that $\frac{2\pi}{2^{n-1}} < \delta$ for any given $\delta > 0$,

So per dense in S^1.

Theorem

The map $g: S^1 \to S^1$, $g(z) = z^2$ is transitive map.

proof Let J_1 and J_2 be any two nonempty open sets (arcs) in S^1,

$$J_1 = (\theta_a, \theta_b), \quad g(\theta) = 2\theta$$

So, $g^2(\theta) = 2^2\theta \quad \ldots \quad g^k(\theta) = 2^k\theta$

$$g^k(J_1) = (2^k\theta_a, 2^k\theta_b)$$

$$2^k\theta_b - 2^k\theta_a$$
$$= 2^k(\theta_b - \theta_a)$$

So we can take K such that

$2^K(\theta_b - \theta_a) > 2\pi$ for some $K \in \mathbb{Z}^+$

i.e $g^K(J_1)$ covers all S^1 So it covers J_2

$g^K(J_1) \cap J_2 \neq \emptyset$

g is transitive map.

Sensitive Dependence

Def

A map of an interval I is said to be posses sensitive dependence on initial conditions if there exists $V > 0$ such that for any $x_0 \in I$ and $\delta > 0$, there exists $y_0 \in (x_0 - \delta, x_0 + \delta)$ and a positive integer k such that

$$\left| f^k(x_0) - f^k(y_0) \right| \geq V$$

The number V will be called the sensitivity constant of f

Example: Consider the doubling map $D: [0,1] \to [0,1]$
Show that D has sensitive dependence on inital conditions.

$$D(x) = 2x \pmod 1 = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}) = I_1 \\ 2x - 1 & \text{if } x \in [\frac{1}{2}, 1) = I_2 \\ 0 & \text{if } x = 1 \end{cases}$$

For any $x, y \in [0,1)$ we have two cases either (i) $x, y \in I$ or (ii) $x \in I_1, y \in I_2$

Case (i) if $x, y \in I_1$ then

$$|D(x) - D(y)| = |2x - 2y| = 2|x - y|$$

on the other hand if $x, y \in I_2$ then

$$|D(x) - D(y)| = |2x - 1 - (2y - 1)| = |2x - 2y| = 2|x - y|$$

Case(ii) Suppose now that $x \in I_1$ and $y \in I_2$. Then

$$|D(x) - D(y)| = |2x - (2y-1)| = |2x - 2y + 1| \geq 1 - |2x - 2y|$$

$$\therefore |D(x) - D(y)| \geq 1 - 2|x - y|$$

if $|x - y| < \frac{1}{4}$

$$|D(x) - D(y)| \geq 1 - \frac{2}{4} = \frac{1}{2}$$

Select $V = \frac{1}{4}$ where V is the sensitivity constant of D.

If $x_0 \in I$ and $\delta > 0$,
we pick any point $y_0 \in (x_0 - \delta, x_0 + \delta)$, if $x_0, y_0 \in I_1$ or I_2
then $|D^k(x_0) - D^k(y_0)| = 2^k |x_0 - y_0| \geq \frac{1}{4}$ for some $k \in \mathbb{Z}^+$

on the other hand if $x_0 \in I_1$ and $y_0 \in I_2$ and $|x_0 - y_0| < \frac{1}{4}$ then

$$|D(x_0) - D(y_0)| = \frac{1}{2} > \frac{1}{4} = V$$

Finally if $x_0 = 0$ or $x_0 = 1$, the above argument will carry over in a straight forward manner.

We have D is SDIC with $V = \frac{1}{4}$.

Def

A map $f: I \to I$ is said to be expansive if there exist $\epsilon > 0$ such that for any $x, y \in I$ with $x \neq y$, there exists $k \in \mathbb{Z}^+$ with

$$|f^k(x) - f^k(y)| > \epsilon$$

clear that f expansive $\longrightarrow$ f is SDIC

Ex: $f(x) = 2x$ on $\mathbb{R}$ is expansive

(Liapunov exponent λ)

The Liapunov exponent $\lambda(x)$ at a point x measures the growth in error per iteration or the average loss of information during successive iterates of point near x.

$$\lambda(x_0) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln |f'(x_k)|$$

(the rate of convergence of two orbits)

where $x_k = f^k(x_0)$

Remark

If the Liapunov exponent λ is positive, then sensitive dependence exists.

Ex Find the Liapunov exponent of the tent map

$$T(x) = \begin{cases} 2x & \text{for } 0 \leq x \leq \frac{1}{2} \\ 2(1-x) & \text{for } \frac{1}{2} < x \leq 1 \end{cases}$$

We have $|T'(x)| = 2 \cdot \forall x \in [0,1] - \{\frac{1}{2}\}$.

$\therefore \lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln(2) = \lim_{n\to\infty} \frac{1}{n} \ln 2^n$

[since $\underbrace{\ln 2 + \ln 2 + \cdots + \ln 2}_{n\text{-times}} = \ln(\underbrace{2 \cdot 2 \cdots 2}_{n\text{-times}})$]

$\lambda(x_0) = \lim_{n\to\infty} \frac{n}{n} \ln 2$

So $\lambda(x_0) = \ln 2 = 0.693 > 0$

Thus T is SDIC

Example

Consider the rotation map on the unit circle S^1 defined by $R_\lambda(\theta) = \theta + 2\pi\lambda$ for some $\lambda \in R$.

$R'_\lambda(\theta) = 1$

$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln 1 = 0$

This clearly indicates that the rotation map R_λ fails to be sensitive

Example

Show that for $1 < M \leq 3$ and $M \neq 2$, the Liapunov exponent of the logistic map $h_M(x) = Mx(1-x)$ is given by $\lambda(x) = \ln|2-M|$

Sol

$L_M(x) = Mx(1-x)$, $p_1 = \frac{M-1}{M} = 1 - \frac{1}{M}$ is attracting

$L'_M(x) = M - 2Mx$

$= M(1-2x)$

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln |L'(x_k)|$$

$$= \lim_{n\to\infty} \left(\frac{1}{n} \sum_{k=0}^{n-1} \ln [M|1-2x_k|] \right), \quad x_k = L_M^k(x_0)$$

We know that L_M has an attracting fixed point

$p_1 = 1 - \frac{1}{M}$ for $(1 < M < 3)$

i.e $x_k \longrightarrow p_1$ as $k \to \infty$

So $|L'_M(x_k)| \longrightarrow |L'_M(p_1)| = M\left|1 - 2\left(1 - \frac{1}{M}\right)\right|$

$= |2 - M|$

Now $\ln |L'(x_k)| \longrightarrow \ln|2-M|$, $M \neq 2$

$$\therefore \lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln|2-M|$$

$$= \lim_{n\to\infty} \frac{1}{n} \ln |2-M|^n$$

$$\therefore \lambda(x_0) = \lim_{n\to\infty} \frac{n}{n} \ln|2-M|$$

$$= \ln|2-M|$$

Definition of Chaos

Def A map $f: I \to I$, where I is an interval, is said to be chaotic if:

1- f is transitive
2- The set of periodic points p is dense in I
3- f has sensitive dependence on initial conditions

Remark: Recently, [Banks et al. 1992] showed that conditions (1) and (2) in above definition imply condition (3) of sensitive dependence on intial conditions.

Theorem

Let $f: I \to I$ be a continuous map on an interval I. If f is transitive and its set of periodic points is dense, then f possesses sensitive dependence on initial conditions, i.e., f is chaotic.

Examples of chaotic maps

1- The Doubling map $D(x) = 2x \pmod 1$ on $[0,1]$

2- The Tent map $T(x) = \begin{cases} 2x, & \text{if } 0 \le x \le \frac{1}{2} \\ 2(1-x), & \text{if } \frac{1}{2} < x \le 1 \end{cases}$

3- The logistic map $L_4(x) = 4x(1-x)$ on $[0,1]$

4- Angle doubling map $g(z) = z^2$ on S^1, $g(\theta) = 2\theta$

5- $L_M(x) = Mx(1-x)$ on Λ, $M > 4$

6- The left Shift map $\sigma: \sum_2^+ \to \sum_2^+$

Cantor Sets

Cantor sets play an important role in both analysis and in chaos theory.
We begin by presenting some notations from topology.

Def

(i) Let A be a subset of $\mathbb{R}$. Then $x_0 \in \mathbb{R}$ is said to be a limit point of A if for every $\delta > 0$, there exists $a \in A \cap (x_0 - \delta, x_0 + \delta)$, $a \neq x_0$.

A is said to be perfect if every point in A is a limit point of A.

(ii) A subset A of $\mathbb{R}$ is connected if it is not the union of two nonempty open subset of $\mathbb{R}$.
Equivalently, $A \subseteq \mathbb{R}$ is connected if and only if it is an interval.

A is said to be totally disconnected if the only nonempty connected subset of A are the one-point sets.

(iii) A subset A of $\mathbb{R}$ is a Cantor set if it is closed, bounded, perfect and totally disconnected.

Example The Cantor Middle-Third Set

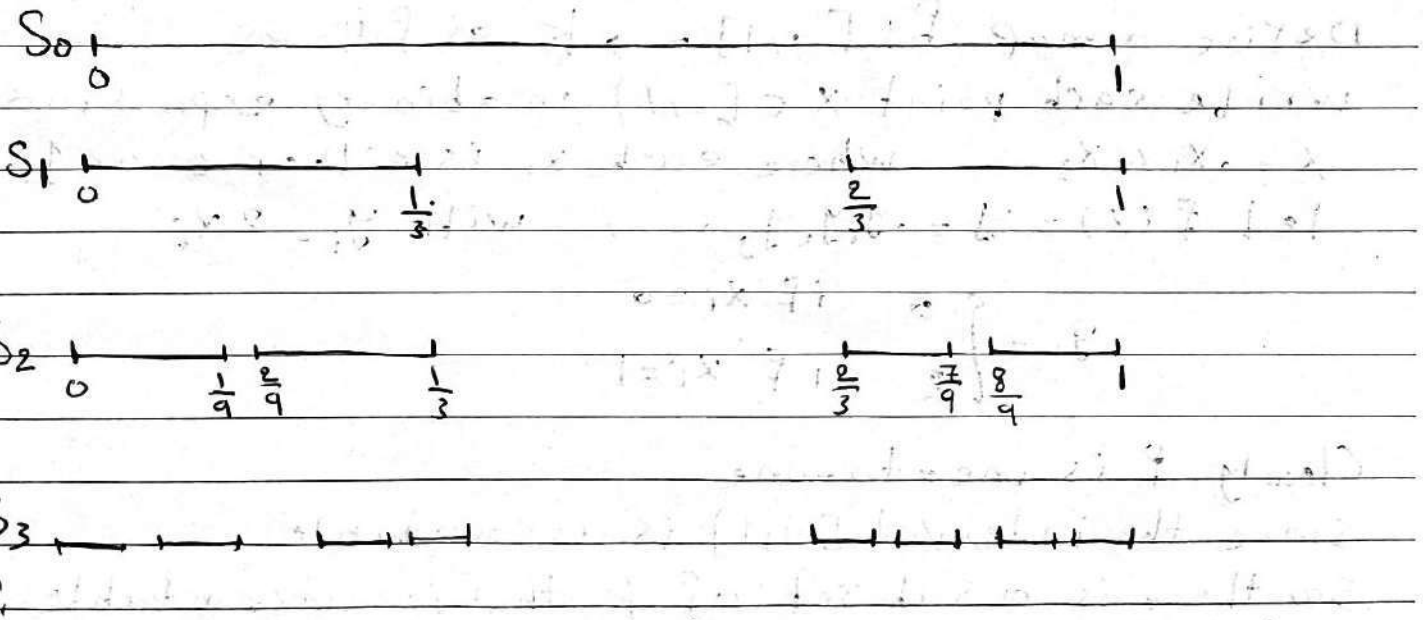
We begin with a bounded closed interval $S_0 = [0,1]$. Remove its open middle third $(\frac{1}{3}, \frac{2}{3})$ and denote the remaining set as S_1 so that $S_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Next remove the open middle third of each of the subintervals of S_1 to obtain $S_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$.

Continue this inductively to obtain a nested sequence of closed intervals $\{S_n\}$, where $S_{n+1} \subset S_n$.

The Cantor set is defined as

$$K = \bigcap_{n=1}^{\infty} S_n$$

Clearly K is nonempty since it contains the end points of the subintervals of each S_n such as $0, \frac{1}{9}, \frac{2}{9}, \frac{1}{3}, \frac{2}{3}, \frac{7}{9}, \ldots$. Furthermore, K is closed since it is the intersection of closed sets, it is also bounded since it is a subset of the interval $[0,1]$.



Ternary Representation of K

When a number has both terminating and nonterminating ternary representation, we agree to use the nonterminating form except when the terminating representation ends in 2.

For example, $\frac{1}{3}$ has two ternary representations, $.1$ and $.0\bar{2}$, but we choose the nonterminating representation $.0\bar{2}$

one may show that $\frac{7}{9} = .20\bar{2}$

$$\frac{1}{9} = .00\bar{2}$$

Theorem

$x \in K$ if and only if it has the ternary representation $.x_1 x_2 x_3 \ldots$, where each x_i is either 0 or 2.

Theorem The Cantor's middle-Thirds set K is uncountable.

Proof

Define a map $f: [0,1] \longrightarrow K$ as follows

write each point $x \in [0,1]$ in a binary expansion $x = .x_1 x_2 x_3 \ldots$ where each x_i is either 0 or 1.

let $f(x) = y = .y_1 y_2 y_3 \ldots$, with $y_i = 2x_i$

$$y_i = \begin{cases} 0 & \text{if } x_i = 0 \\ 2 & \text{if } x_i = 1 \end{cases}$$

Clearly f is one-to-one

since the interval $[0,1]$ is uncountable

So, there is a subset of K that is uncountable

So K it self is uncountable.

Sets of Measure Zero

Def Let I be a bounded subinterval of $\mathbb{R}$ (finite interval), I having end points a, b $(a \leq b)$. The length of I is then $|I| = b - a$.

If unbounded interval, we set $|I| = \infty$

for example $\mathbb{R}$ is a set of length ∞

Def A subset $A \subseteq \mathbb{R}$ is a set of measure zero if we can cover A by bounded countable open intervals, so that the total length of intervals can be chosen to be arbitrary small.

i.e given any $\epsilon > 0$, there is a collection of open intervals $\{ I_n : n \in \mathbb{Z}^+ \}$ with $A \subseteq \bigcup_{n=1}^{\infty} I_n$ and

$$\sum_{n=1}^{\infty} |I_n| \leq \epsilon$$

Examples: $\mathbb{K}$, $\mathbb{Q}$ (rational no's), any finite set

Def A subset $A \subseteq \mathbb{R}$ is countable if $\exists f : A \to \mathbb{Z}^+$ one-to-one map.

Examples of Countable sets

$\mathbb{Q}, \mathbb{Z}, \mathbb{N}, \mathbb{Z}^-$
any finite set

Examples of uncountable sets

$\mathbb{R}, \mathbb{Q}^c$

Symbolic Dynamics

We will now introduce the sequence space $\sum_2^+$ of all the one-sided sequences on the two symbols 0 and 1. This is defined as follows

$$\sum_2^+ = \{ x = \{x_n\}_{n=0}^{\infty} \; ; \; x_n = 0 \text{ or } 1 \}.$$

Thus, elements of $\sum_2^+$ are infinite strings of 0's and 1's such as

$\{0\,0\,1\,0\,1 \ldots\}$, $\{0\,1\,0\,0 \ldots\}$ etc.

An overbar over a group of digits indicates that the group is repeated indefinitely. For example

$$\{0\,1\,\overline{0\,0\,1\,0}\ldots\}$$

We define the shift map $\sigma : \sum_2^+ \longrightarrow \sum_2^+$ by letting

$$\sigma\{x_0, x_1 x_2 x_3 \ldots\} = \{x_1 x_2 x_3 \ldots\}.$$

for example, $\sigma\{1\,0\,1\,\bar{1} \ldots\} = \{0\,1\,\bar{1} \ldots\}$

There are two fixed points of σ:
The constant sequences $\{0\,\bar{0} \ldots\}$ and $\{1\,\bar{1} \ldots\}$

The K-periodic points are of the form $\{\overline{x_0 x_1 \ldots x_{k-1}} \ldots\}$.

Since there are 2^k ways of arranging the block $x_0 x_1 \ldots x_{k-1}$, it follows that there are 2^k periodic points of period k.

Next, we define a metric (or a distance function) on Σ_2^+ as follows:

for $x = \{x_n\}$ and $y = \{y_n\}$ in Σ_2^+, we let

$$d(x,y) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i}$$

Remark

1- Since $|x_i - y_i|$ is either 0 or 1

we have $d(x,y) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i} \leq \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{(\frac{1}{2})^0}{1 - \frac{1}{2}} = 2$

Hence, this metric is bounded.

2- Suppose that $x_i = y_i$ for $i = 0, 1, 2, \ldots, n$. Then

$d(x,y) \leq \frac{1}{2^n}$. To show this, observe that

$$d(x,y) = \sum_{i=0}^{n} \frac{0}{2^i} + \sum_{i=n+1}^{\infty} \frac{|x_i - y_i|}{2^i}$$

$$\leq \sum_{i=n+1}^{\infty} \frac{1}{2^i}$$

Thus $d(x,y) \leq \frac{1}{2^n}$.

3- If, on the other hand, $d(x,y) < \frac{1}{2^n}$, then $x_i = y_i$ for $i = 0, 1, 2, \ldots, n$

To show this, assume $x_j \neq y_j$ for some $j \leq n$.

Then $d(x,y) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i} \geq \frac{1}{2^j} \geq \frac{1}{2^n}$ a contradiction.

Based on the above remarks, the following conclusions

can be made.

Lemma

The function $d(x,y) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i}$ is a metric on Σ_2^+.

Example

Find $d(x,y)$ if

1- $x = \{0\,1\,\bar{1} \dots\}$ and $y = \{1\,0\,\bar{1} \dots\}$

2- $x = \{0\,1\,0\,1\,0\,1\,\bar{0}\}$ and $y = \{0\,1\,0\,1\,0\,1\,\bar{1} \dots\}$

Sol

1- $d(x,y) = \frac{|x_1 - y_1|}{2^0} + \frac{|x_2 - y_2|}{2} + \dots = 1 + \frac{1}{2} + 0 + \dots = \frac{3}{2}$

2- $d(x,y) = \sum_{i=6}^{\infty} \frac{1}{2^i} = \frac{\frac{1}{2^6}}{1 - \frac{1}{2}} = \frac{1}{2^5}$

Lemma The shift map $\sigma : \Sigma_2^+ \longrightarrow \Sigma_2^+$ is continuous.

Theorem

The shift map $\sigma : \Sigma_2^+ \longrightarrow \Sigma_2^+$ is chaotic on Σ_2^+.

Proof We first prove that the set of periodic points of σ is dense in Σ_2^+ (i.e to prove σ is a regular system)

So let $x = \{x_i\} = \{x_0, \dots x_n, \dots\}$ be an arbitrary point in Σ_2^+.

Define $y = (\overline{x_0\, x_1\, x_2 \dots x_n})$ which is a periodic point.

$$d(x,y) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i} = \sum_{i=n+1}^{\infty} \frac{\bar{x}_i}{2^i} \leq \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{(\frac{1}{2})^{n+1}}{1 - \frac{1}{2}} = \frac{1}{2^n}$$

$\therefore d(x,y) \leq \frac{1}{2^n} \longrightarrow 0$ as $n \longrightarrow \infty$

So σ is regular system.

Next, we need to prove σ is a transitive map (a dense orbit in $\sum_2^+$)

Consider the sequence

z: 0 1 , 00 01 10 11 , 000 001 010 100 011 101 110 111
1-block 2-block, 3-blocks

This sequence consists of all the blocks of 0's and 1's of length 1 (there are only 2), of length 2 (there are four 2-blocks), etc.

We want to show that $O(z)$ dense in $\sum_2^+$ (i.e $\overline{O(z)} = \sum_2^+$)

let $x \in \sum_2^+$

Choose $n \in Z^+$ such that
the first $n+1$ terms $x_0 x_1 x_2 \cdots x_n$ in the sequence x must appear in the n-blocks in the sequence z.

There fore, $\exists\, k \in Z^+$, $\sigma^k(z)$ agrees with x in the first n terms

$$d(\sigma^k(z), x) = \sum_{i=n+1}^{\infty} \frac{\overline{x_i}}{2^i} \leq \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{(\frac{1}{2})^{n+1}}{1-\frac{1}{2}} = \frac{1}{2^n}$$

$\therefore d(\sigma^k(z), x) \leq \frac{1}{2^n} \longrightarrow 0$ as $n \longrightarrow \infty$
and $\sigma^k(z) \in O(z)$ (by def. of orbit)
So σ is transitive map
Thus σ is chaotic.

Remark $\sum_2^+$ is a Cantor set

Define a map $h: K \longrightarrow \sum_2^+$ as follows:

For $x = 0.x_0 x_1 x_2 \ldots$ with x_i is either 0 or 2, we let

$$h(x) = y = \{y_0, y_1, y_2, \ldots\}, \text{ where } y_i = \frac{x_i}{2}$$

It is not hard to verify that h is a homeomorphism, that is, h is one-to-one, onto, continuous such that h^{-1} has the same properties

Thus, $\sum_2^+$ inherits all of the topological properties of K

Thus $\sum_2^+$ is a Cantor Set.

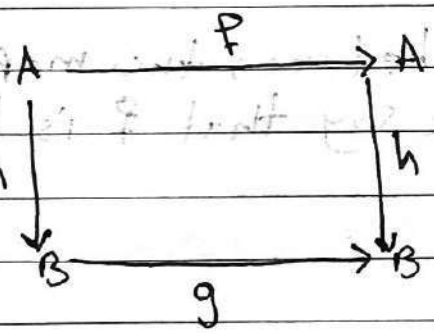
Def

If $f: J \longrightarrow J$ is a one-to-one and onto map such that f and f^{-1} are continuous, then f is said to be a homeomorphism

For example the map $f(x) = x^3$ is a homeomorphism on $\mathbb{R}$.
whereas the map $f(x) = x^2$ is not a homeomorphism on $\mathbb{R}$ but a homeomorphism on $\mathbb{R}^+$

Conjugacy

Def Let $f: A \to A$ and $g: B \to B$ be given maps. Then f and g are said to be conjugate, denoted by $f \simeq g$, if there exists a homeomorphism $h: A \to B$ such that $h \circ f = g \circ h$



The map f is h conjugate to the map g.

Example Show that $L_4(x) = 4x(1-x)$ on $[0,1]$ is linearly conjugate to the map $f(x) = 2x^2 - 1$ on $[-1,1]$

$h(x) = -2x + 1$

Solution

$$\begin{array}{ccc} L_4: [0,1] & \longrightarrow & [0,1] \\ \downarrow & & \downarrow h(x) = -2x+1 \\ f: [-1,1] & \longrightarrow & [-1,1] \end{array}$$

Clear that $h: [0,1] \longrightarrow [-1,1]$ be a linear map

So, h is linear map and h is homeomorphism

Now $h \circ L_4(x) = h(4x(1-x))$

$= -2[4x(1-x)] + 1$

$= -8x + 8x^2 + 1$

and $f \circ h(x) = f(-2x+1)$

$= 2(-2x+1)^2 - 1$

$= 2[4x^2 - 4x + 1] - 1$

$$= 8x^2 - 8x + 1$$

$\therefore h \circ L_4 = f \circ h$

Thus $L_4 \underset{h}{\approx} f$

Def

If the conjugacy map h between two maps f and g happens to be linear, we say that f is linearly conjugate to g.

Example

Consider the logistic map $L_M(x) = Mx(1-x)$ with $0 < M \leq 4$ and the quadratic map $G(x) = ax^2 + bx + c$, where $a \neq 0$. Show that L_M and G are conjugate via the homeomorphism

$$h(x) = -\frac{M}{a}x + \frac{M-b}{2a}$$

Solution

It is easy to verify that the map h is homeomorphism

Now if

$h(L_M(x)) = G(h(x))$, we must have

$$-\frac{M}{a}[Mx(1-x)] + \frac{M-b}{2a} = a\left[-\frac{M}{a}x + \frac{M-b}{2a}\right]^2 +$$

$$b\left(-\frac{M}{a}x + \frac{M-b}{2a}\right) + c$$

which gives

$$c = \frac{b^2 - M^2 + 2M - 2b}{4a}$$

Therefore, for this value of c, L_M is h-conjugate to G

properties of Conjugate maps

1- Show that if $f \approx g$, then $f^n \approx g^n$ for all $n \in \mathbb{Z}^+$

Proof

To prove $f^n \approx g^n$ (i.e to prove $h \circ f^n = g^n \circ h$ $\forall n \in \mathbb{Z}^+$

Since $f \approx g$ then $h \circ f = g \circ h$

So $h \circ f^n(x) = h(f^n(x)) = h(f(f^{n-1}(x)))$

$= g(h(f^{n-1}(x)))$ [since $h(f(x)) = g(h(x))$]

$= g(h(f(f^{n-2}(x))))$

$= g(g(h(f^{n-2}(x))))$

$= g^2(h(f^{n-2}(x))) = \cdots$

$= g^n(h(x)) = g^n \circ h(x)$

$\therefore f^n \approx g^n$

2- If $f: A \to A$ and $g: B \to B$ and $f \approx g$ and

(i) if p is a point of period k for f then $h(p)$ is a point of period k for g.

(ii) f is one-to one if and only if g is one-to-one

(iii) f is onto if and only if g is onto.

Show that the conjugacy relation $\approx$ between maps is an equivalence relation

Proof

We have $f \underset{h}{\approx} g$ i.e $h \circ f(x) = g \circ h(x)$

1- To prove $\approx$ is symmetry relation (i.e $g \underset{h^{-1}}{\approx} f$)

$h^{-1}(h(f(x))) = h^{-1}(g(h(x)))$

$i(f(x)) = h^{-1}(g(h(x)))$ [since $h^{-1} \circ h = i$]

$f(x) = h^{-1} \circ g \circ h(x)$

$f \circ h^{-1}(x) = h^{-1} \circ g \circ h \circ h^{-1}(x)$

$\therefore f \circ h^{-1}(x) = h^{-1} \circ g \circ i(x)$

Thus $h^{-1} \circ g = f \circ h^{-1} \Rightarrow g \underset{h^{-1}}{\approx} f$

2- To prove $\approx$ is reflexive relation.

Clear that $i \circ f = f \circ i$

where $i : A \to A$ the identity map so $f \underset{i}{\approx} f$

3- To prove $\approx$ is transitive relation

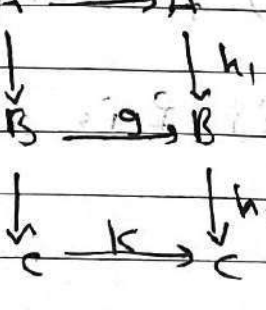
i.e if $f \underset{h_1}{\approx} g$ and $g \underset{h_2}{\approx} k$ then $f \approx k$

We know $h_1 \circ f = g \circ h_1$ ..... (1)

and $h_2 \circ g = k \circ h_2$ ..... (2)

T.P $(h_2 \circ h_1) \circ f = k \circ (h_2 \circ h_1)$

and $h_2 \circ h_1$ is homeomorphism



$$(h_2 \circ h_1) \circ f = h_2 \circ (h_1 \circ f)$$
$$= h_2 (g \circ h_1) \quad \text{by (1)}$$
$$= (h_2 \circ g) \circ h_1$$
$$= (k \circ h_2) \circ h_1 \quad \text{by (2)}$$
$$= k \circ (h_2 \circ h_1)$$

and since both h_1, h_2 are homeomorphism then so as $h_2 \circ h_1$.

Example consider $f: (0,\infty) \longrightarrow (0,\infty)$, $f(x) = 2x$
and $g: R \longrightarrow R$, $g(x) = x + \ln 2$
and $h: (0,\infty) \longrightarrow R$, $h(x) = \ln x$
show that: $f \approx g$.

Sol

$$h \circ f(x) = h(2x) = \ln(2x)$$
$$g \circ h(x) = g(\ln x) = \ln x + \ln 2$$
$$= \ln(2x)$$

$\therefore h \circ f = g \circ h$ and h is homeomorphism.
$\ln x$ (1-1 & onto, h, h^{-1} cont.)

$\therefore f \approx g$

$$\begin{array}{ccc} (0,\infty) & \xrightarrow{f(x)=2x} & (0,\infty) \\ \downarrow h & & \downarrow h(x)=\ln x \\ R & \xrightarrow{g(x)=x+\ln 2} & R \end{array}$$

f is SDIC, but g not SDIC

Theorem

Suppose that the map $f: A \to A$ is h-conjugate to the map $g: B \to B$. Then f is chaotic on A, if and only if g is chaotic on B.

Proof

Suppose that f is chaotic and $f \underset{h}{\approx} g$.

To show that g is chaotic, we first show that g is regular system (i.e has a dense set of periodic points)

Let $U \subseteq B$ be non-empty open

$h^{-1}(U)$ open in A [h is continuous]

$h^{-1}(U)$ contains periodic points of f. [f is regular]

Say $x \in A$ which is k-periodic point, $x \in h^{-1}(U)$

$$x = f^k(x) \longrightarrow h(x) = h(f^k(x)) \quad [h \text{ is map}]$$
$$= g^k(h(x)) \quad [f^k \underset{h}{\approx} g^k]$$

So $h(x)$ is k-periodic point of g

$h(x) \in h(h^{-1}(U)) = U$

Thus g is regular system.

Next, we show that g is transitive map.

Suppose $O(z) = \{z, f(z), f^2(z), \ldots\}$ is dense in A

Let $V \subseteq B$ be a non-empty open set

then $h^{-1}(V)$ is open non-empty in A [h is continuous]

So $O(z) \cap h^{-1}(V) \neq \phi$; [f is transitive map]

i.e $\exists k \in \mathbb{Z}^+$ s.t $f^k(z) \in h^{-1}(V)$

$\therefore h(f^k(z)) = g^k(h(z)) \in V \quad [f^k \underset{h}{\simeq} g^k]$

and $g^k(h(z)) \in O(h(z)) = \{h(z), g(h(z)), g^2(h(z)), \ldots$ is dense in β. i.e g is transitive
Thus g is chaotic

Similary, if g is chaotic then f is chaotic $[g \underset{h^{-1}}{\simeq} f]$

Def The map $f: A \to A$ and $g: \beta \to \beta$ are said to be semiconjugate if there exists a map $h: A \to \beta$ with $h \circ f = g \circ h$ and such that h is onto, continuous, and open.

Theorem

If $f: A \to A$ and $g: \beta \to \beta$ are semiconjugate then f is chaotic, if and only if g is chaotic.

Example

Use the logistic map L_4 to show that the Tent map T is chaotic on $I = [0,1]$
Define the map $h: I \to I$ by letting $h(x) = \sin^2(\frac{\pi}{2}x)$
Then h semiconjugates T with L_4.
To prove this, we with show that $L_4 \circ h = h \circ T$

Now $L_4 \circ h = L_4(h(x)) = 4\sin^2(\frac{\pi}{2}x)[1 - \sin^2(\frac{\pi}{2}x)]$
$= 4\sin^2(\frac{\pi}{2}x)\cos^2(\frac{\pi}{2}x)$
$= \sin^2(\pi x)$

on the other hand

$$h(T(x)) = \begin{cases} h(2x) = \sin^2(\pi x) & \text{if } 0 \le x \le \frac{1}{2} \\ h(2-2x) = \sin^2(\pi - \pi x) = \sin^2 \pi x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

Thus h semiconjugates T with L_4

and hence T is chaotic

Example

Use the double-angle map $g: S^1 \to S^1$, $g(\theta) = 2\theta$ (which is chaotic), to show that $L_4(x) = 4x(1-x)$ is chaotic on $[0,1]$.

Sol

Define $h: S^1 \to [0,1]$, $h(\theta) = \sin^2 \frac{\theta}{2}$

not 1-1, onto, continuous and open

$$h \circ g(\theta) = h(g(\theta)) = \sin^2 \theta$$
$$= L_4(h(\theta))$$

Thus h is semiconjugates g with L_4

Since g is chaotic on S^1 it follows that L_4 is chaotic on I.

Stability of Two-Dimensional Maps

Linear Maps Vs. Linear Systems

Recall from linear algebra that a map $L: \mathbb{R}^2 \to \mathbb{R}^2$ is called a linear transformation if

1- $L(U_1 + U_2) = L(U_1) + L(U_2)$ for $U_1, U_2 \in \mathbb{R}^2$

2- $L(\alpha U) = \alpha L(U)$ for $U \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$

Moreover, it is always possible to represent L (with given basis for $\mathbb{R}^2$) by a matrix A.

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

which may be written in the form

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

or $$L(U) = AU \qquad (1)$$

where $U = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

By iterating L, we conclude that $L^n(U) = A^n U$.

Hence, the orbit of U under L is given by

$$\{U, AU, A^2U, \ldots, A^nU, \ldots\} \qquad (2)$$

Thus, to compute the orbit of U, it suffices to compute A^nU for $n \in \mathbb{Z}^+$

Another way of looking at the same problem is by considering the following two-dimensional system of difference equations

$$x(n+1) = a\,x(n) + b\,y(n) \qquad (3)$$
$$y(n+1) = c\,x(n) + d\,y(n)$$

or

$$U(n+1) = A\,U(n) \qquad (4)$$

By iteration, one may show that the solution of equation (4) is given by

$$U(n) = A^n U(0).$$

So, if we let $U_0 = U(0)$, then $L^n(U_0) = U(n)$

The form of equation (3) is more convenient when we are considering applications in biology, engineering, economics and so forth. For example $x(n)$ and $y(n)$ may represent the population sizes at time period n of two competitive species or preys and predators.

Computing A^n

Consider a matrix $A = (a_{ij})$ of order 2×2. Then: $P(\lambda) = \det(A - \lambda I)$ is called the characteristic polynomial of A and its zeros are called the eigenvalues of A.

Associated with each eigenvalue λ of A a nonzero eigenvector $V \in R^2$ with $AV = \lambda V$.

Remark

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the characteristic equation of A is

$$\lambda^2 - tr(A)\lambda + |A| = 0, \text{ where } tr(A) = a + d$$

$$tr(A) = \lambda_1 + \lambda_2 = T$$

$$|A| = \lambda_1 \lambda_2 = D$$

where λ_1, λ_2 are the eigenvalues of A.

Example Find the eigenvalues and the eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

Solution

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix} = 0 \longrightarrow (2-\lambda)(4-\lambda) - 3 = 0$$

$$\therefore \lambda^2 - 6\lambda + 5 = 0 \longrightarrow (\lambda - 1)(\lambda - 5) = 0$$

$$\therefore \lambda_1 = 1, \quad \lambda_2 = 5$$

To find the correspondic eigenvector V_1, we solve the vector equation $AV_1 = \lambda V_1$ or $(A - \lambda_1 I)V_1 = 0$

For $\lambda_1 = 1$

$(A - \lambda_1 I) V_1 = 0$

$$\left[\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} - (1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} V_{11} \\ V_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence $V_{11} + 3V_{21} = 0 \longrightarrow V_{11} = -3V_{21}$

let $V_{21} = 1 \longrightarrow V_{11} = -3 \Longrightarrow V_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

For $\lambda_2 = 5$, $(A - \lambda_2 I) V_2 = 0$

$$\begin{pmatrix} 2-5 & 3 \\ 1 & 4-5 \end{pmatrix} \begin{pmatrix} V_{11} \\ V_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} V_{11} \\ V_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow V_{11} = V_{21}$$

let $V_{11} = 1 \longrightarrow V_{21} = 1 \longrightarrow V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Def The matrices A and B are said to be similar if there exists a nonsingular matrix P such that

$AP = PB$, i.e, $P^{-1} AP = B$

We note here that the relation "similarity" between matrices is an equivalence relation, i.e

1- A is similar to A

2- If A is similar to B then B is similar to A.

3- If A is similar to B and B is similar to C, then A is similar to C.

Theorem

Let A and B be two similar matrices. Then A and B have the same eigenvalues.

Proof

$$A \approx B \longrightarrow P^{-1}AP = B \Rightarrow A = PBP^{-1}$$

Let λ be an eigenvalue of A

$$AV = \lambda V$$

$$PBP^{-1}V = \lambda V$$

Hence $BP^{-1}V = \lambda P^{-1}V$

$\therefore \lambda$ is an eigenvalue of B with $P^{-1}V$ as eigenvector

Theorem

Let A be 2x2 real matrix. Then A is similar to one of the following matrices

① $J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $\lambda_1 \neq \lambda_2$ real,

$$J^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \quad , P = (V_1 \; V_2)$$

② $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, $\lambda_1 = \lambda_2 = \lambda$ real

$$J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix} \quad , P = (V_1 \; V_2)$$

Solve $(A - \lambda I) V_2 = V_1$ for V_2

③ $J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ $\lambda_1 = \alpha + i\beta$ (complex)
$\lambda_2 = \overline{\lambda_1} = \alpha - i\beta$

$$J^n = |\lambda_1|^n \begin{pmatrix} \cos nw & \sin nw \\ -\sin nw & \cos nw \end{pmatrix}$$

$w = \arg \lambda_1 = \tan^{-1} \frac{\beta}{\alpha}$, $P = (V_1 \; V_2)$

$V_1 = Re(V)$, $V_2 = Im(V)$
V eigenVector with λ_1

$$\boxed{A^n = P J^n P^{-1}}$$

Example Solve the system of difference equations

$$X(n+1) = AX(n)$$

where

$$A = \begin{pmatrix} -4 & 9 \\ -4 & 8 \end{pmatrix}, \quad X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Solution

$$|A - \lambda I| = 0 \longrightarrow \begin{vmatrix} -4-\lambda & 9 \\ -4 & 8-\lambda \end{vmatrix} = 0 \quad \text{or}$$

$$\lambda^2 - tr(A)\lambda + |A| = 0 \longrightarrow \lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda - 2)^2 = 0 \longrightarrow \lambda = 2 \quad \text{So}$$

$$A \approx J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

$$(A - \lambda I) V_1 = 0$$

$$\begin{pmatrix} -6 & 9 \\ -4 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \longrightarrow 3y - 2x = 0$$

$$\therefore y = \frac{2}{3}x \text{, let } x = 3 \longrightarrow y = 2 \longrightarrow V_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Now $(A - \lambda I) V_2 = V_1$

$$\begin{pmatrix} -6 & 9 \\ -4 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \longrightarrow 3y - 2x = 1 \text{, let } x$$

$$\therefore V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{So} \quad P = (V_1 \; V_2) = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$$

$$P^{-1} = \frac{1}{|P|} \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

$$\therefore X_n = A^n X_0 = P J^n P^{-1} X_0$$

$$= \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2^n & n 2^{n-1} \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2^n - n 2^n \\ -2^{n+1} \end{pmatrix}$$

$$= 2^n \begin{pmatrix} 3(1-n) - 2 \\ 2(1-n) - 2 \end{pmatrix}$$

$$= 2^n \begin{pmatrix} 1 - 3n \\ -2n \end{pmatrix}$$

Example Solve the system of difference equations

$$X(n+1) = AX(n) \quad , \quad X(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

where

$$A = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix}$$

Solution The characteristic equation of A is

$$\lambda^2 - T\lambda + D = 0 \quad , T = -4 \quad , D = 13$$

$$\therefore \lambda^2 + 4\lambda + 13 = 0 \rightarrow \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda = \frac{-4 \pm \sqrt{T^2 - 4D}}{2} = -2 \pm \frac{\sqrt{16 - 4(13)}}{2}$$

$$= -2 \pm \frac{\sqrt{36}\, i}{2} = -2 \pm 3i$$

$\therefore \lambda_1 = -2 + 3i \quad , \quad \lambda_2 = -2 - 3i$

$|\lambda_1| = \sqrt{13} \qquad \alpha = -2 \quad , \beta = 3$

$\therefore J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \approx A$

$J = \begin{pmatrix} -2 & 3 \\ -3 & -2 \end{pmatrix} \approx A$

$J^n = |\lambda_1|^n \begin{pmatrix} \cos nw & \sin nw \\ -\sin nw & \cos nw \end{pmatrix} \quad , w = \tan^{-1} -\frac{3}{2}$

$P = (Re(V) \quad Im(V))$

To find V solve $(A - \lambda_1 I) V = 0$

$$\left[\begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} - (-2+3i) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2+2-3i & -3 \\ 3 & -2+2-3i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore -3ix - 3y = 0$

let $x = -1 \rightarrow y = i$, So $V = \begin{pmatrix} -1 \\ i \end{pmatrix}$

$Re(V) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad , \quad Im(V) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\therefore P = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad P^{-1} = \frac{1}{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\therefore X_n = A^n X_0 = P J^n P^{-1} X_0$$

$$= (13)^{\frac{n}{2}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos nw & \sin nw \\ -\sin nw & \cos nw \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= (13)^{\frac{n}{2}} \begin{pmatrix} -\cos nw & -\sin nw \\ -\sin nw & \cos nw \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$= (13)^{\frac{n}{2}} \begin{pmatrix} -\cos nw - 2\sin nw \\ \sin nw + 2\cos nw \end{pmatrix}$$

Stability of Linear Systems

Theorem

Let $X_{n+1} = A X_n$ be a system of difference equations then

① If $\rho(A) < 1$, then the origin is asymptotically stable

② If $\rho(A) > 1$, then the origin is unstable

③ If $\rho(A) = 1$, then the origin is unstable if the Jordan form is $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ (i.e $\lambda_1 = \lambda_2$) and stable otherwise

spectral radius of A defined as $\rho(A) = \max\{|\lambda_1|, |\lambda_2| : \lambda_1, \lambda_2$ are the eigenvalues of $A\}$

$$\rho(A) = \max_i \{|\lambda_i|\}$$

Examples

① $A = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix}$, $\rho(A) = \sqrt{13} > 1$

$\therefore \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is unstable

② $A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{-4}{5} \end{pmatrix}$, $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{-4}{5}$

$\rho(A) = \frac{4}{5} < 1$ $\therefore \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is asymptotically stable

Theorem

Let $A = (a_{ij})$ be a 2×2 matrix. Then $\rho(A) < 1$

if and only if $|\text{tr } A| - 1 < \det A < 1$

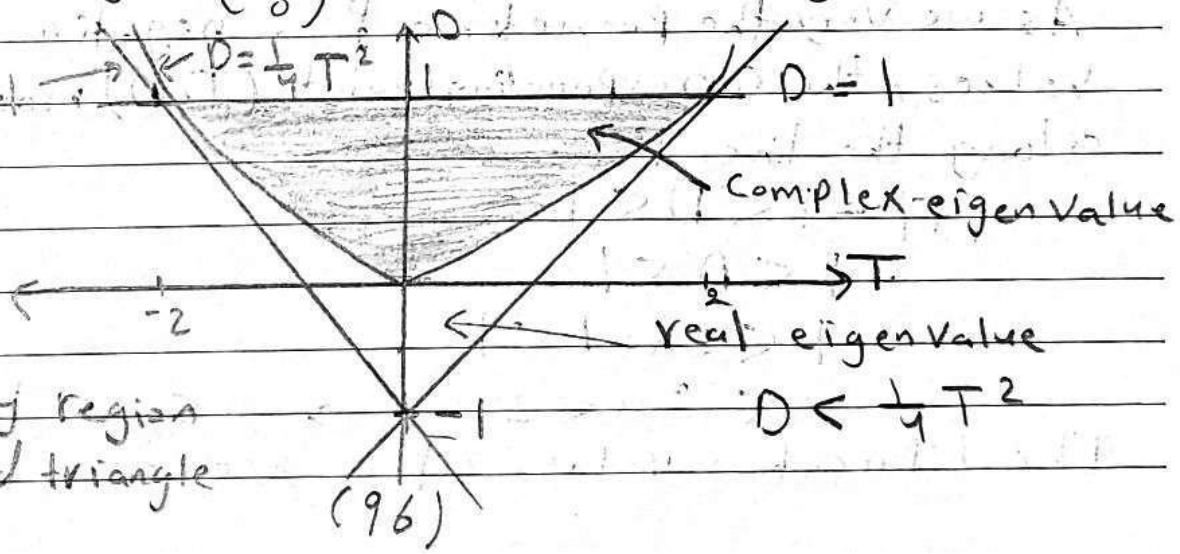
Example

Let $X_{n+1} = A X_n$, where $\text{tr}(A) = \frac{-5}{4}$ and $|A| = \frac{1}{2}$, then

$|T| - 1 = \frac{1}{4} < D = \frac{1}{2} < 1$

then $\rho(A) < 1$

So the origin $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is asymptotically stable.



The stability region is the shaded triangle

Navigating the Trace-Determinant Plane

The trace-determinant plane is effective in the study of linear systems with parameters.
There are three critical loci. Let T denotes the trace and D denote the determinant. Then there are three critical lines: $D = trA - 1$, $D = -trA - 1$, $D = 1$ they enclose the stability region in the trace-determinant planes.

We now illustrate our analysis by the following example.

Consider the one-parameter family of linear systems $X(n+1) = A\,X(n)$, where

$A = \begin{pmatrix} -1 & a \\ -2 & 1 \end{pmatrix}$. Find the bifurcation values of a.
(the values of a where critical dynamical changes occur).

$T = tr(A) = 0$
$D = |A| = 2a - 1$

As we vary the parameter a from negative to positive values, the corresponding point (T, D) moves vertically along the line $T = 0$

$|T| - 1 < D < 1$
$-1 < D < 1$
$-1 < 2a - 1 < 1$
$0 < 2a < 2 \rightarrow 0 < a < 1$

The bifurcation values of a are $a = 0$, $a = 1$

Remark If $a=0$ then $A = \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}$

To find the fixed points in this case

$L(X) = X \longrightarrow AX = X$

$\begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow x=0$ which is the y-axis (every point on the y-axis is a fixed point of L

To find the periodic points of period 2

$A^2X = X \quad (L^2(X) = X)$

$\begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow$ all points in the plane are period-2 points except the y-axis points

Liapunov Functions for Nonlinear Maps

In 1892, the Russian mathematician A. M. Liapunov (sometimes transliterated as Lyapunov) introduced a new method to investigate the stability of nonlinear differential equations. This method, now known as Liapunov's second method, allows one to determine the stability of solutions to a differential equation without actually solving it.

Def A real valued function $V: G \to R$, $G \subseteq R^2$, is said to be a Liapunov function on G if

① V is continuous on G

② $V(x) \geq 0 \quad \forall x \in G$

③ $\Delta V(x) \leq 0$, whenever x and $f(x) \in G$

where $\Delta V(x) = V(f(x)) - V(x)$

Theorem

Suppose that V is Liapunov function on a neighborhood G of a fixed point x^* of a continuous map f on R^2. Then x^* is stable. If, in addition,
$\Delta V(x) < 0 \quad \forall x$ and $f(x) \in G - \{x^*\}$
then x^* is asymptotically stable.

Example Consider the two-dimensional system

$$x(n+1) = \frac{a\,y(n)}{1+x^2(n)}$$

$$y(n+1) = \frac{b\,x(n)}{1+y^2(n)}$$

or the map

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ay/(1+x^2) \\ bx/(1+y^2) \end{pmatrix}$$

Discuss the stability of the zero solution of the equations

Solution

Take $V(x,y) = x^2+y^2$, then $V(x) \geq 0$, and V is continuous on R^2

$$\Delta V\begin{pmatrix} x \\ y \end{pmatrix} = V\left(f\begin{pmatrix} x \\ y \end{pmatrix}\right) - V\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \frac{a^2y^2}{(1+x^2)^2} + \frac{b^2x^2}{(1+y^2)^2} - x^2 - y^2$$

$$= \left(\frac{b^2}{(1+y^2)^2} - 1\right)x^2 + \left(\frac{a^2}{(1+x^2)^2} - 1\right)y^2$$

$$\leq (b^2-1)x^2 + (a^2-1)y^2$$

Now, we have three cases

case ① if $a^2 < 1$ and $b^2 < 1$, then

$\Delta V(x) < 0 \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is asymptotically stable

case ② if $a^2 \leq 1$ and $b^2 \leq 1$, then

$\Delta V(x) \leq 0 \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is stable

case ③ if $a^2 > 1$ and $b^2 > 1$, then the theorem fails (since V not Liapunov function in this case).

Stability via linearization

For the nonlinear map $f: R^2 \to R^2$ we can define the Jacobian matrix

$$Df(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} \quad \text{where}$$

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

Now if x^* is a fixed point of f, then:

$$A = Df(x^*) = \left. \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} \right|_{x^*}$$

Theorem

Let $f: G \subset R^2 \to R^2$ be a C^1 map, where G is an open subset of R^2, x^* is a fixed point of f, and $A = Df(x^*)$. Then the following statements hold true:

1- If $\rho(A) < 1$, then x^* is asymptotically stable.

2- If $\rho(A) > 1$, then x^* is unstable.

3- If $\rho(A) = 1$, then x^* may or may not be stable.

The Hénon Map

Our aim is limited to the discussion of stability and attractors of the Hénon map, given by

$$x(n+1) = 1 - a x^2(n) + y(n)$$
$$y(n+1) = b\, x(n)$$

where a and b are real parameters with $|b| < 1$.

For fixed points $H(X) = X$

$$\begin{pmatrix} 1 - ax^2 + y \\ bx \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\rightarrow \quad 1 - ax^2 + y = x$$
$$bx = y$$

$$\rightarrow 1 - ax^2 + bx = x \rightarrow ax^2 + (1-b)x - 1 = 0$$

$$\therefore x = \frac{b-1 \pm \sqrt{(1-b)^2 + 4a}}{2a}$$

So $\begin{pmatrix} x_1^* \\ y_1^* \end{pmatrix} = \begin{pmatrix} \frac{1}{2a}(b-1+\sqrt{(1-b)^2+4a} \\ \frac{b}{2a}(b-1+\sqrt{(1-b)^2+4a} \end{pmatrix} = X_1^*$ and

$$\begin{pmatrix} x_2^* \\ y_2^* \end{pmatrix} = \begin{pmatrix} \frac{1}{2a}(b-1-\sqrt{(1-b)^2+4a} \\ \frac{b}{2a}(b-1-\sqrt{(1-b)^2+4a} \end{pmatrix} = X_2^*$$

$$DH(x) = \begin{pmatrix} \frac{\partial H_1}{\partial x} & \frac{\partial H_1}{\partial y} \\ \frac{\partial H_2}{\partial x} & \frac{\partial H_2}{\partial y} \end{pmatrix}$$

$$= \begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix}$$

$$|\det(DH(x))| = |-b| < 1$$

the H is Contraction

H^n the Hénon attractor.